3-Algebras and 3D Superconformal Chern-Simons-Matter Theories

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M-theory is the underlying theory of five different string theories and 11D supergravity theory.

While strings (1+1D) are fundamental objects in string theory, M2-branes (1+2D) are fundamental objects in M-theory.

According to the Gauge/Gravity duality, a gravity theory is equivalent to a gauge theory.

Extended $(\mathcal{N} \geq 4)^1$ superconformal Chern-Simons-matter (CSM) theories in 3D are natural candidates of the dual gauge theories of multi M2-branes.

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$^1$Here ‘$\mathcal{N}$’ stands for $\mathcal{N}$ copies of supersymmetries. In 3D, if $\mathcal{N} = 1$, there are two independent fermionic generators.
For example, M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$, with $k$ the Chern-Simons level, is equivalent to an $\mathcal{N} = 6, U(N) \times U(N)$ superconformal CSM (ABJM) theory in 3D (Aharony et al. (2008) [1]). These two theories have the same amount of supersymmetries.

The $\mathcal{N} = 4, 5, 6$ CSM theories were constructed by using ordinary Lie 2-algebras\(^2\).

On the other hand, (1) the $\mathcal{N} = 8$ (BLG) theory was constructed in terms of Nambu 3-algebra (Bagger and Lambert (2008) [2], and Gustavsson (2008) [3]), where the Nambu 3-bracket is defined by

$$[t^a, t^b, t^c] = f^{abc} dt^d, \quad (1)$$

and (2) the $\mathcal{N} = 6$ ABJM was re-constructed in terms of Hermitian 3-algebra (Bagger and Lambert (2008) [7]).

\(^2\)The $\mathcal{N} = 4$ GW theory, Gaiotto and Witten (2008) [4]; The $\mathcal{N} = 4$ quiver gauge theory, Hosomichi et al. (2008) [5]; The general $\mathcal{N} = 5, 6$ theories, Hosomichi et al. (2008) [6].
• However, it remains unclear whether these theories can be constructed in a unified 3-algebra approach or not.

• It is also natural to ask whether there are new examples of the extended superconformal CSM theories.

We propose to solve these two problems: (1) We introduce the notion of symplectic 3-algebra, and (2) demonstrate that the $D = 3$, $\mathcal{N} = 4,5,6,8$ CSM theories are constructed in terms of this unified 3-algebraic structure, and (3) show that taking advantage of symplectic 3-algebra, some new $\mathcal{N} = 4$ quiver gauge theories can be derived as well. (Chen and Wu (2010), [8], Chen (2010), [9]; Chen and Wu (2010), [10]; Chen and Wu (2013), [11].)
II. Symplectic 3-algebras

A 3-algebra is a complex vector space equipped a 3-bracket, mapping three vectors to one vector (Chen and Wu (2009) [8]):

\[
[T_I, T_J; T_K] = f_{IJK}^L T_L, \quad (2)
\]

\[
([M^m, M^n] = C^{mn}_p M^p). \quad (3)
\]

where \( T_I \) \((I = 1, 2, \cdots, M)\) is a set of generators. The set of complex numbers \( f_{IJK}^L = f_{JIK}^L \) are called the structure constants.

There is a realization in terms of a double grade commutator of a super Lie algebra

\[
[T_I, T_J; T_K] \doteq [\{Q_I, Q_J\}, Q_K], \quad (4)
\]

where \( Q_I \) are a set of fermionic generators.

We define the global transformation of a field \( X \)
valued in this 3-algebra as (Bagger, et al. (2008),[2]):

$$\delta \tilde{\Lambda} X = \Lambda^{IJ}[T_I, T_J; X], \quad X = X^K T_K \quad (5)$$

$$\delta \tilde{\lambda} \mathcal{O} = \lambda^m [M_m, \mathcal{O}], \quad \mathcal{O} = \mathcal{O}^n M_n. \quad (6)$$

where the parameter $\Lambda^{IJ}$ is independent of spacetime coordinate.

These is a fundamental identity (FI) satisfied by the generators:

$$[T_I, T_J; [T_M, T_N; T_K]] = [[T_I, T_J; T_M], T_N; T_K] \quad (6)$$

$$+ [T_M, [T_I, T_J; T_N]; T_K] + [T_M, T_N; [T_I, T_J; T_K]].$$

In other words, the doubled 3-brackets are constrained by the FI. The 3-algebra can be viewed as a generalization of the ordinary (Lie) 2-algebra, with the FI plays the role of ‘Jacobi identity’.

While the ordinary Jacobi Identity is equivalent to the statement

$$\delta \tilde{\lambda} ([M_m, \mathcal{O}]) = [\delta \tilde{\lambda} M_m, \mathcal{O}] + [M_m, \delta \tilde{\lambda} \mathcal{O}], \quad (7)$$
the FI of the 3-algebra can be derived by requiring that

\[ \delta_{\tilde{\Lambda}}([T_I, T_J; X]) = [\delta_{\tilde{\Lambda}} T_I, T_J; X] + [T_I, \delta_{\tilde{\Lambda}} T_J; X] \]
\[ + [T_I, T_J; \delta_{\tilde{\Lambda}} X]. \quad (8) \]

Also, the transform (5) is required to preserve both the anti-symmetric form \( \omega(X, Y) = \omega_{IJ} X^I Y^J \), and the Hermitian form \( h(X, Y) = X^I Y^I \) simultaneously (Chen, Wu (2009) [8]):

\[ \delta_{\tilde{\Lambda}} \omega(X, Y) = \delta_{\tilde{\Lambda}} h(X, Y) = 0. \quad (9) \]

Also, to close the extended (3D) Poincare superalgebra, the structure constants must satisfy the linear constraint equation \( f_{(IJK)L} = 0 \). We will use \( \omega_{IJ}, f_{IJKL} \) and \( h(X, Y) \) to construct the superconformal CSM theories.
1. Gauging the symmetry

We assume that the $\mathcal{N} = 1$ superfields for the matter fields are 3-algebra valued (Chen, Wu (2010) [10]):

$$
\Phi^I_A(x, \theta) = Z^I_A(x) + i \theta \gamma^B A \psi^I_B(x) - \frac{i}{2} \theta^2 F^I_A(x),
$$

where $I$ is a 3-algebra index, $A, B$ are $Sp(4) \cong SO(5)$ indices ($A, B = 1, \ldots, 4$).

To construct the $\mathcal{N} = 1$ CSM theory, we first gauge the global symmetry transformation (5). We define the gauge transformation of the superfield $\Phi^I$ as

$$
\delta_{\Lambda} \Phi_A = \Lambda^{KL} [T_K, T_L; \Phi_A]
$$

(11)

or

$$
\delta_{\Lambda} \Phi^I_A = \Lambda^{KL} f_{KL}^I J \Phi^J_A = \hat{\Lambda}^I J \Phi^J_A,
$$

(12)
where the parameter $\Lambda^{KL}$ is a superfield, depending on the coordinates of the superspace, i.e. $\Lambda^{KL} = \Lambda^{KL}(x, \theta)$.

We then define the covariant derivatives as

$$
(D_\alpha)^I_J = \mathcal{D}_\alpha \delta^I_J + \tilde{\Gamma}_\alpha^I_J, \quad (D_\mu)^I_J = \partial_\mu \delta^I_J + \tilde{\Gamma}_\mu^I_J,
$$

(13)

where $\mathcal{D}_\alpha$ is the super-covariant derivative, and $\tilde{\Gamma}_\alpha^I_J$ is the super-connection. In accordance with our basic definition (5), it is natural to assume that the super-connections take the following forms

$$
\tilde{\Gamma}_\alpha^I_J \equiv \Gamma_{\alpha}^{KL} f_{KL}^I_J \quad \text{and} \quad \tilde{\Gamma}_\mu^I_J \equiv \Gamma_{\mu}^{KL} f_{KL}^I_J,
$$

(14)

Actually, imposing the conventional constraint (Gates et al. (1983), [14])

$$
\{D_\alpha, D_\beta\} = 2i D_{\alpha\beta} = 2i \gamma^\mu_{\alpha\beta} D_\mu
$$

(15)

determines the vector superconnection (the second equation of (14)).

To be self-consistent, the covariant derivative $D_\alpha$
must satisfy the Jacobi identity:

\[
[D_\alpha, \{D_\beta, D_\gamma\}] + [D_\beta, \{D_\gamma, D_\alpha\}] + [D_\gamma, \{D_\alpha, D_\beta\}] = 0.
\]

(16)

The Jacobi identity can be solved by introducing a superfield strength $\tilde{\mathcal{W}}_\alpha$ (Chen, Wu (2010) [10]).

2. Constructing the Action of the $\mathcal{N} = 5$ Theory

We use the super-fields (10), the covariant derivatives (13), the super-connections (14) and the superfield strength $\tilde{\mathcal{W}}_\alpha$ to construct a $\mathcal{N} = 1$ CMS theory, then enhance the $\mathcal{N} = 1$ super-symmetry to $\mathcal{N} = 5$ by imposing the linear constraint equation $f_{(IJK)L} = 0$. In summary, the $\mathcal{N} = 5$ Lagrangian in
terms of the symplectic 3-algebra is given by

\[ \mathcal{L} = \frac{1}{2}(-D_\mu \bar{Z}_I^A D^\mu Z_A^I + i\bar{\psi}_I^A \gamma_\mu D^\mu \psi_A^I) \]

\[ -\frac{i}{2} \omega^{AB} \omega^{CD} f_{IJKL} (Z_A^I Z_B^K \psi_C^J \psi_D^L - 2Z_A^I Z_D^K \psi_C^J \psi_B^L) \]

\[ + \frac{1}{2} \epsilon^{\mu \nu \lambda} (f_{IJKL} A_\mu^{IJ} \partial_\nu A_\lambda^{KL}) \]

\[ + 2 f_{IJK}^O f_{OLMN} A_\mu^{IJ} A_\nu^{KL} A_\lambda^{MN} \]

\[ + \frac{1}{60} (2f_{IJK}^O f_{OLMN} - 9f_{KLI}^O f_{ONMJ}) \]

\[ + 2f_{IJK}^O f_{OKMN}) Z_N^A Z_A^I Z_J^B Z_K^L Z_M^C. \]

The potential is \textbf{sixth order} in the scalar fields \( Z \). The covariant derivatives are given by

\[ D_\mu Z_I^A = \partial_\mu Z_I^A - \tilde{A}_\mu^J I Z_J^A, \quad D_\mu Z_A^I = \partial_\mu Z_A^I + \tilde{A}_\mu^I J Z_J^A. \]

(17)

And the \( \mathcal{N} = 5 \) supersymmetry transformations are
the following:

\[
\begin{align*}
\delta Z^I_A &= i \epsilon^B_A \psi^I_B, \\
\delta \psi^I_A &= \gamma^\mu D_\mu Z^I_B \epsilon^B_A + \frac{1}{3} f^{IJK} \omega^{BC} Z^J_B Z^K_C Z^L_D \epsilon^D_A \\
&\quad - \frac{2}{3} f^{IJK} \omega^{BD} Z^J_C Z^K_D Z^L_A \epsilon^C_B, \\
\delta \tilde{A}_\mu^K_L &= i \epsilon^{AB} \gamma_\mu \psi^I_B Z^I_A f_{IJK}^L. 
\end{align*}
\]

(18)

Here \( \epsilon^B_A \) is the parameter; and \( \gamma^\mu \) (\( \mu = 0, 1, 2 \)) are the 3D gamma matrices. The (on-shell) closure of the above \( \mathcal{N} = 5 \) SUSY transformations are verified explicitly (Chen, (2009),[9]).
IV. Three-Algebras and Super Lie Algebras

In the $\mathcal{N} = 5$ case, we use the following simple superalgebra

\[
\begin{align*}
[M^m, M^n] &= C^{mn} M^s, \\
[M^m, Q_I] &= -\tau^m_{IJ} \omega^{JK} Q_K, \\
\{Q_I, Q_J\} &= \tau^m_{IJ} k_{mn} M^n, \quad (19)
\end{align*}
\]

to realize the symplectic 3-algebra in the $\mathcal{N} = 5$ theory. Here $I = 1, \cdots, 2L$, and $\omega^{JK} = -\omega^{KJ}$ and $k_{mn}$ are invariant quadratic forms on the superalgebra.

The key idea of the superalgebra realization of 3-algebra is to identify the 3-algebra generators $T_I$ with the superalgebra generators $Q_I$, and to construct the 3-bracket in terms of a double graded commutator on the superalgebra, i.e.,

\[
\begin{align*}
T_I &\equiv Q_I, \quad (20) \\
[T_I, T_J; T_K] &\equiv [\{Q_I, Q_J\}, Q_K] = k_{mn} \tau^m_{IJ} \tau^n_L Q_L.
\end{align*}
\]
In this realization, the structure constants are nothing but the tensor product,

\[ f_{IJK}^L = k_{mn} \tau_{IJ}^m \tau_{KL}^n. \]  

(21)

Also, in this realization, the FI of the 3-algebra can be converted into the \( MMQ \) Jacobi identity of the superalgebra, and the constraint equation \( f(IJK)_L = 0 \) for enhancing the \( \mathcal{N} = 1 \) supersymmetry to \( \mathcal{N} = 5 \) is equivalent to the \( QQQ \) Jacobi identity of the superalgebra.

The resulting Lie algebra of the gauge group is just the bosonic subalgebra of the superalgebra (19), and the corresponding representation is determined by the fermionic generators.

For instance, if we use \( U(M|N) \), whose bosonic subalgebra is \( U(M) \times U(N) \), then the matter fields are in the bi-fundamental representation of \( U(M) \times U(N) \).

Similarly, we can use superalgebras to realize the 3-algebras in \( \mathcal{N} = 4, 6, 8 \) theories.
V. $\mathcal{N} = 4$ Theories and 3-Algebras

We have been able to constructed the $\mathcal{N} = 4$ quiver gauge theory in terms of a symplectic 3-algebra (Chen, Wu (2010) [10]). This 3-algebra contains two symplectic sub 3-algebras, whose generators are $T_a$ and $T_a'$, respectively. Except for their own 3-brackets

$$[T_a, T_b; T_c] = f_{abc}^dT_d, \quad [T_a', T_b'; T_c'] = f_{a'b'c'}^{d'}T_{d'},$$

(22)

there are two non-trivial 3-brackets between $T_a$ and $T_a'$:

$$[T_a, T_b; T_c'] = f_{abc}^{d'}T_{d'}, \quad [T_a', T_b'; T_c] = f_{a'b'c}^dT_d.$$

(23)

So this 3-algebra includes four sets of 3-brackets and four sets of fundamental identities (FI’s). The structure constants of these four 3-brackets are required to satisfy certain symmetry and reality conditions, and two linear constraint equations, $f_{(abc)}d = f_{(a'b'c')}d' = 0$, for closing the $\mathcal{N} = 4$ super Poincare algebra.

The un-twisted multiplets $(Z^a_A, \psi^a_A)$ are valued in

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the sub 3-algebra, where $A, \dot{A} = 1, 2$ transform in the two-dimensional representation of the $SU(2) \times SU(2)$ R-symmetry group. And the twisted multiplets $(Z_{A}'^a, \psi_A^a)$ are valued in another sub 3-algebra.

The action and the law of supersymmetry transformations are very complicated (See Chen and Wu (2013), [11]); We do not present them here.

As for the $\mathcal{N} = 4$ case, we use two superalgebras $G$ and $G'$ to construct the two sub symplectic 3-algebras, respectively:

$$T_a \dot{=} Q_a, \quad T_a' \dot{=} Q_a',
$$

$$[T_a, T_b; T_c] \dot{=} [\{Q_a, Q_b\}, Q_c] = f_{abc}^d Q_d,$$

$$[T_{a'}, T_{b'}; T_{c'}] \dot{=} [\{Q_{a'}, Q_{b'}\}, Q_{c'}] = f_{a'b'c'}^{d'} Q_{d'}.$$

The two non-trivial 3-brackets between $T_a$ and $T_a'$ (23)
can be constructed as follows

\[
[T_a, T_b; T'_c] = \{Q_a, Q_b\} = \{Q_{a'}, Q_{b'}\} = f_{abc}^d Q_d,
\]

\[
[T'_a, T'_b; T'_c] = \{Q_{a'}, Q_{b'}\} = f_{a'b'c} d Q_d.
\]

The bosonic parts of $G$ and $G'$ share at least one simple factor or $U(1)$ factor is a sufficient condition for that $\{Q_a, Q_b\}, Q_{c'} \neq 0$ and $\{Q_{a'}, Q_{b'}\}, Q_c \neq 0$.

In this realization, the Lie algebra of the gauge group of the $\mathcal{N} = 4$ quiver gauge theory is the bosonic subalgebras of $G$ and $G'$. The corresponding representation is determined by the fermionic generators $Q_a$ and $Q_{b'}$.

For instance, if we choose $G = OSp(2|2N)$ and $G' = U(M|1)$ whose common bosonic factor is $SO(2) \cong U(1)$, then the resulting gauge group is $Sp(2N) \times U(1) \times U(M)$, and the un-twisted multiplet is in the bi-fundamental representation of $USp(2N) \times U(1)$, while the twisted multiplet is in the bi-fundamental representation of $U(1) \times U(M)$. This is a new class of $\mathcal{N} = 4$ quiver CSM gauge theories.
More new classes of $\mathcal{N} = 4$ quiver CSM gauge theories can be found in our work (Chen and Wu (JHEP, 2013), [11]).
VI. Discussions

- It would be nice to construct the gravity duals of the $\mathcal{N} = 4$ quiver CSM gauge theories. (Most of them are not found yet.)

- And it would be interesting to re-derive these CSM theories by using brane constructions.

- Wilson loops? Integrability? (For 3D $\mathcal{N} = 4$ quiver CSM gauge theories.)

- Three-algebras may have connections with 4D conventional QFT.
THANK YOU VERY MUCH!
References


