Noval Half-BPS Wilson Loops in 3d $\mathcal{N}=4$ Chern-Simons-matter theories

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Outline

• Background – classification of BPS Wilson loops in CSM theories

• M2-branes in AdS$_4$ *S$^7$/ Γ

• Constructions of half-BPS Wilson loops in 3d N=4 CSM theories

• Discussions
Backgrounds

• The **classification** of BPS Wilson loops (WLs) in 3d super Chern-Simons theories is more complicated than similar problems in 4d SYM.

• There are two types: **Gaiotto-Yin** type and **Drukker-Trancanelli** type.

• The construction of DT type WLs involves fermions.
Backgrounds

• GY type WLs exist in CSM theories with at least $N=2$ SUSY.
• DT type WLs exist for $N=5, 6$ CSM theories. [Drukker etal 09][Lee, Lee, 10]
• We have strong evidence that BPS WLs can be at most 1/3 BPS. Either DT type WL does not exist here or it can be at most 1/3 BPS. [Chen, JW, Zhu, 14]
• How about the case of $N=4$ theories?
Reviews

• ABJM theory is 3d $N=6$ Chern-Simons-matter theory with gauge group $U(N)^*U(N)$ and Chern-Simons level $(k, -k)$.

• The scalar fields are four complex scalars in the bifundamental representation $\left( N, \bar{N} \right)$

(N1=N2=N in the fig.)
Supersymmetry transformation

\[ \delta A_\mu = \frac{4\pi}{k} \left( \phi_I \bar{\psi}_J \gamma_\mu \theta^{IJ} + \bar{\theta}_{IJ} \gamma_\mu \psi^J \phi^I \right), \]

\[ \delta \hat{A}_\mu = \frac{4\pi}{k} \left( \bar{\psi}_J \gamma_\mu \phi_I \theta^{IJ} + \bar{\theta}_{IJ} \phi^I \gamma_\mu \psi^J \right), \]

\[ \delta \phi_I = 2i \bar{\theta}_{IJ} \psi^J, \quad \delta \phi^I = 2i \bar{\psi}_J \theta^{IJ}, \]

\[ \delta \psi^I = 2 \gamma^\mu \theta^{IJ} D_\mu \phi_J - \frac{4\pi}{k} \theta^{IJ} \left( \phi_J \phi^K \phi_K - \phi_K \phi^K \phi_J \right) - \frac{8\pi}{k} \theta^{KL} \phi_K \phi^I \phi^L, \]

\[ \delta \bar{\psi}_I = -2 \bar{\theta}_{IJ} \gamma^\mu D_\mu \phi^J + \frac{4\pi}{k} \bar{\theta}_{IJ} \left( \phi^J \phi_K \phi^K - \phi^K \phi_K \phi^J \right) + \frac{8\pi}{k} \bar{\theta}_{KL} \phi_K \phi_I \phi^L. \]
GY type Wilson loops in ABJM theory

• Following GY, we consider the WLs (before take the trace)

\[ W = \mathcal{P} e^{-i \int A}, \]
\[ A = A_0 + \frac{2\pi}{k} M^{I J} \phi_I \phi^J. \]

• \( \delta A=0 \) gives only 1/6-BPS WLs with \( M=\text{diag}(1, 1, -1, -1) \).
• \( \text{SU}(4)_R \) is broken to \( \text{SU}(2)^* \text{SU}(2)^* \text{U}(1) \).
• Non-constant \( M \) leads to less supersymmetry (general rule).

• [Drukker, Plefka, Young 08] [Chen, JW, 08]
• [Rey, Suyama, Yamaguchi, 08]
Gravity dual

• The simplest string solution dual to WL is $\text{AdS}_2$ in $\text{AdS}_4$ and which is a point in $\mathbb{CP}^3$.

• This solution is half-BPS and the isometry of $\mathbb{CP}^3$ is broken to $\text{SU}(3) \times \text{U}(1)$.

• So the immediate problem is to construct the half-BPS WLs in ABJM theory.
Half-BPS Wilson loops

• In 2009, Drukker and Trancanelli constructed the following WL (we consider the **timelike** Wilson line going from infinite to infinite)

\[ W = \mathcal{P} e^{-i \int L}, \]

\[ L = \begin{pmatrix} \mathcal{A} & f_1 \\ \bar{f}_2 & \hat{\mathcal{A}} \end{pmatrix}. \]

\[ \mathcal{A} = A_0 + \frac{2\pi}{k} M^I J \phi_I \phi^J, \]

\[ \hat{\mathcal{A}} = \hat{A}_0 + \frac{2\pi}{k} N_I J \phi^I \phi_J, \]

\[ f_1 = \sqrt{\frac{2\pi}{k}} \bar{\eta}_I \psi^I, \]

\[ \bar{f}_2 = \sqrt{\frac{2\pi}{k}} \bar{\psi} \eta^I. \]
Half-BPS Wilson loops

• We only need $\delta L = D_0 G \equiv \partial_0 G + i[L, G]$, for the WL to be BPS.

• Recall that though the gauge potential is not gauge covariant, the WL is gauge invariant.
Parameters for half-BPS WL

- \( M=N=\text{diag}(-1, 1, 1, 1) \)

\[
\bar{\eta}_I = \bar{\eta} \delta^I_1, \quad \eta^I = \eta \delta^I_1,
\]

\[
\gamma_0 \eta = i \eta, \quad \bar{\eta} \gamma_0 = i \bar{\eta}.
\]

\[
\eta \bar{\eta} = -i - \gamma_0,
\]
M2-branes in AdS$_4*S^7/\Gamma$
Set-up

• We put multi-membranes at $C^4/\Gamma$, where the discrete group $\Gamma$ is generated by

\[
(z_1, z_2, z_3, z_4) \to (\omega_m z_1, \omega_m z_2, z_3, z_4),
\]

\[
(z_1, z_2, z_3, z_4) \to (\omega_m k z_1, \omega_m k z_2, \omega_m k z_3, \omega_m k z_4),
\]

\[
\omega_n \equiv \exp(2i\pi/n)
\]

• The near horizon limit gives M-theory on $\text{AdS}_4 \ast S^7/\Gamma$. This background is dual to 3d $N=4$ SCFT.
M2-brane solution

• The simplest M2 brane dual to WL has topology AdS$_2$*$S^1$.
• AdS$_2$ part is inside AdS$_4$. $S^1$ part is inside $S^7/\Gamma$ and is along the M-theory circle.

• After some computations, we found that this probe M2-brane is half-BPS w. r. t. the background.

• So there should be half-BPS WL in the dual field theory.
Half BPS Wilson Loops in 3d $N=4$ Chern-Simons-matter theories
Dual field theory

- The conformal field theory dual to M-theory on AdS$_4$*$S^7$/Γ can be obtained from non-chiral orbifold of ABJM theory. The obtained theory has $N=4$ supersymmetries. [Benna, Klebanov, Klose, Smedback, 08].
Supersymmetry transformation

\[ \delta A^{(2\ell+1)}_\mu = \frac{4\pi}{k} \left[ \left( \phi_i^{(2\ell+1)} \psi_i^{(2\ell+1)} - \phi_i^{(2\ell)} \psi_i^{(2\ell)} \right) \gamma_\mu \theta^i + \bar{\theta}_{i\ell} \gamma_\mu \left( \psi_i^{(2\ell+1)} \bar{\phi}_i^{(2\ell+1)} - \psi_i^{(2\ell)} \bar{\phi}_i^{(2\ell)} \right) \right], \]

\[ \delta \hat{A}^{(2\ell)}_\mu = \frac{4\pi}{k} \left[ \left( \psi_i^{(2\ell-1)} \phi_i^{(2\ell-1)} - \psi_i^{(2\ell)} \phi_i^{(2\ell)} \right) \gamma_\mu \theta^i + \bar{\theta}_{i\ell} \gamma_\mu \left( \phi_i^{(2\ell-1)} \psi_i^{(2\ell-1)} - \phi_i^{(2\ell)} \psi_i^{(2\ell)} \right) \right], \]

\[ \delta \phi_i^{(2\ell+1)} = 2i\bar{\theta}_{ii} \psi_i^{(2\ell+1)}, \quad \delta \phi_i^{(2\ell)} = -2i\bar{\theta}_{ii} \psi_i^{(2\ell)}, \]

\[ \delta \bar{\phi}_i^{(2\ell+1)} = 2i\psi_i^{(2\ell+1)} \theta^i, \quad \delta \bar{\phi}_i^{(2\ell)} = -2i\psi_i^{(2\ell)} \theta^i, \]

\[ (\theta^i) = \bar{\theta}_{ii}, \quad \bar{\theta}_{ii} = \epsilon_{ij} \epsilon_{kj} \theta^{kj}. \]
Supersymmetry transformation

\[ \delta \psi^i_{(2\ell)} = 2\gamma^\mu \theta^{ii} D_\mu \phi^i_{(2\ell)} - \frac{4\pi}{k} \theta^{ii} \left( \phi^i_{(2\ell)} \overline{\phi}^{ij}_{(2\ell-1)} \phi^j_{(2\ell)} + \phi^i_{(2\ell)} \phi^{i\bar{j}}_{(2\ell)} \phi^{(2\ell)}_{\bar{j}} \right) \\
- \phi^{(2\ell+1)}_{\bar{j}} \overline{\phi}^{i\bar{j}}_{(2\ell+1)} \phi^i_{(2\ell)} - \phi^{(2\ell)}_{\bar{j}} \phi^{i\bar{j}}_{(2\ell)} \phi^i_{(2\ell+1)} \right) , \]

\[ \delta \psi^i_{(2\ell+1)} = -2\gamma^\mu \theta^{ii} D_\mu \phi^i_{(2\ell+1)} + \frac{4\pi}{k} \theta^{ii} \left( \phi^i_{(2\ell+1)} \overline{\phi}^{ij}_{(2\ell+1)} \phi^j_{(2\ell+1)} + \phi^i_{(2\ell+1)} \phi^{i\bar{j}}_{(2\ell+2)} \phi^{(2\ell+2)}_{\bar{j}} \right) \\
- \phi^{(2\ell+1)}_{\bar{j}} \overline{\phi}^{i\bar{j}}_{(2\ell+1)} \phi^i_{(2\ell+1)} - \phi^{(2\ell)}_{\bar{j}} \phi^{i\bar{j}}_{(2\ell)} \phi^i_{(2\ell+1)} \right) , \]
Supersymmetry transformation

\[ \delta \bar{\psi}_i^{(2\ell)} = -2 \bar{\theta}_{ii} \gamma^\mu D_\mu \bar{\phi}_i^{(2\ell+1)} + \frac{4\pi}{k_c} \bar{\theta}_{ii} \left( \bar{\phi}_i^{(2\ell+1)} \phi_j^{(2\ell+1)} \bar{\phi}_j^{(2\ell)} + \bar{\phi}_i^{(2\ell)} \phi_j^{(2\ell+1)} \bar{\phi}_j^{(2\ell)} - \bar{\phi}_i^{(2\ell)} \phi_j^{(2\ell-1)} \bar{\phi}_j^{(2\ell)} - \bar{\phi}_i^{(2\ell+1)} \phi_j^{(2\ell+1)} \bar{\phi}_j^{(2\ell)} \right) + \frac{8\pi}{k_c} \bar{\theta}_{jj} \left( \bar{\phi}_i^{(2\ell-1)} \phi_j^{(2\ell)} \bar{\phi}_j^{(2\ell)} - \bar{\phi}_i^{(2\ell+1)} \phi_j^{(2\ell+1)} \bar{\phi}_j^{(2\ell+1)} \right), \]

\[ \delta \bar{\psi}_i^{(2\ell+1)} = 2 \bar{\theta}_{ii} \gamma^\mu D_\mu \bar{\phi}_i^{(2\ell+1)} - \frac{4\pi}{k_c} \bar{\theta}_{ii} \left( \bar{\phi}_i^{(2\ell+1)} \phi_j^{(2\ell+1)} \bar{\phi}_j^{(2\ell+1)} + \bar{\phi}_i^{(2\ell+1)} \phi_j^{(2\ell+1)} \bar{\phi}_j^{(2\ell+1)} - \bar{\phi}_i^{(2\ell+1)} \phi_j^{(2\ell+1)} \bar{\phi}_j^{(2\ell+1)} - \bar{\phi}_i^{(2\ell+2)} \phi_j^{(2\ell+2)} \bar{\phi}_j^{(2\ell+1)} \right) + \frac{8\pi}{k_c} \bar{\theta}_{jj} \left( \bar{\phi}_i^{(2\ell+1)} \phi_j^{(2\ell+2)} \bar{\phi}_j^{(2\ell+2)} - \bar{\phi}_i^{(2\ell+2)} \phi_j^{(2\ell+2)} \bar{\phi}_j^{(2\ell+2)} \right). \]
$\frac{1}{4}$-BPS GY type Wilson loops

\[ W^{(2\ell+1)} = \mathcal{P} \exp \left( -i \int \mathcal{A}^{(2\ell+1)} \right), \]

\[ \mathcal{A}^{(2\ell+1)} = A_0^{(2\ell+1)} + \frac{2\pi}{k} \left( M^i_j \phi_{i}^{(2\ell+1)} \bar{\phi}_j^{(2\ell+1)} + M^i_j \phi_i^{(2\ell)} \bar{\phi}_j^{(2\ell)} \right). \]

\[ \hat{W}^{(2\ell)} = \mathcal{P} \exp \left( -i \int \hat{\mathcal{A}}^{(2\ell)} \right), \]

\[ \hat{\mathcal{A}}^{(2\ell)} = \hat{A}_0^{(2\ell)} + \frac{2\pi}{k} \left( N^i_j \phi_{(2\ell-1)} \phi_j^{(2\ell-1)} + N^i_j \phi_i^{(2\ell)} \bar{\phi}_j^{(2\ell)} \right), \]

• All the matrices $M$ and $N$ are diag(-1, 1).
Half-BPS DT type Wilson loops

\[ W = \mathcal{P} e^{-i \int L}, \quad L = \begin{pmatrix} A & F_1 \\ \bar{F}_2 & \hat{A} \end{pmatrix}. \]

\[ A = \begin{pmatrix} A^{(1)} \\ \vdots \\ A^{(2n-1)} \end{pmatrix}, \]

\[ A^{(2\ell+1)} = A^{(2\ell+1)}_0 + \frac{2\pi}{k} \left( M^i_{\bar{j}} \phi^{(2\ell+1)}_i \phi^{(2\ell+1)}_{\bar{j}} + M^i_{\bar{j}} \phi^{(2\ell)}_i \phi^{(2\ell)}_{\bar{j}} \right), \]
Half-BPS WL

\[ \hat{A} = \begin{pmatrix} \hat{A}^{(0)} \\ \hat{A}^{(2)} \\ \vdots \\ \hat{A}^{(2n-2)} \end{pmatrix}, \]

\[ \hat{A}^{(2\ell)} = \hat{A}_0^{(2\ell)} + \frac{2\pi}{k} \left( N_i^j \bar{\phi}_{(2\ell-1)} \phi_j^{(2\ell-1)} + N_i^j \bar{\phi}_{(2\ell)} \phi_j^{(2\ell)} \right), \]
Half-BPS WL

\[ F_1 = \begin{pmatrix}
  f_1^{(0)} & f_1^{(1)} & f_1^{(2)} & f_1^{(3)} & \cdots & \cdots & f_1^{(2n-4)} & f_1^{(2n-3)} & f_1^{(2n-2)} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
  f_1^{(2n-1)} & f_1^{(2n-1)} & \cdots & f_1^{(2n-1)} & \cdots & \cdots & f_1^{(2n-1)} & \cdots & \cdots \\
\end{pmatrix}, \]

\[ f_1^{(2\ell+1)} = \sqrt{\frac{2\pi}{k_i}} \tilde{\eta}_i^{(2\ell+1)} \psi_{(2\ell+1)}, \quad f_1^{(2\ell)} = \sqrt{\frac{2\pi}{k_i}} \tilde{\eta}_i^{(2\ell)} \psi_{(2\ell)}, \]
Half-BPS WL

\[ \tilde{F}_2 = \begin{pmatrix}
\tilde{f}^{(0)}_2 \\
\tilde{f}^{(1)}_2 \\
\tilde{f}^{(2)}_2 \\
\vdots \\
\tilde{f}^{(2n-1)}_2 \\
\end{pmatrix}, \\
\tilde{f}^{(2\ell+1)}_2 = \sqrt{\frac{2\pi}{k_i}} \tilde{\psi}^{(2\ell+1)}_i \eta^{(2\ell+1)}_i, \quad \tilde{f}^{(2\ell)}_2 = \sqrt{\frac{2\pi}{k_i}} \tilde{\psi}^{(2\ell)}_i \eta^{(2\ell)}_i, \]
Half-BPS WL

\[
\begin{align*}
\bar{\eta}_i = \bar{\eta} \delta^1_i, & \quad \eta^i = \eta \delta^i_1, & \quad \bar{\eta} = \eta^i = 0 \\
M^i_j = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, & \quad M^i_j = \begin{pmatrix} m_1 \\ m_1 \end{pmatrix}, \\
N_i^j = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, & \quad N_i^j = \begin{pmatrix} n_1 \\ n_1 \end{pmatrix}.
\end{align*}
\]

\[
m_1 = n_1 = -1, \quad m_2 = m_1 = n_2 = n_1 = 1,
\]

\[
\gamma_0 \eta^{(2\ell)} = i \eta^{(2\ell)}, \quad \bar{\eta}^{(2\ell)} \gamma_0 = i \bar{\eta}^{(2\ell)}, \quad \eta^{(2\ell)} \bar{\eta}^{(2\ell)} = -i I_{2 \times 2} - \gamma_0
\]
Side Remark on a subtle point

• All of these BPS Wilson loops in Minkowski spacetime are time-like.

• There are no space-like BPS Wilson loops due to some real conditions.

• Anybody wrote down this claim clearly?
• (Same results found in von Dennis Muller’s M. Sc. Thesis, but he did not give the clear statement.)

• In Euclidean case, we can relax the real condition and can define the BPS WL.
• Put the theory on $S^3$ and perform the supersymmetry localization.
• Check if the half-BPS DT-type WL is in the same Q-cohomology as special $\frac{1}{4}$-BPS GY-type WL. (Q is the supercharge used for localization).
• If so, compute the vev of half-BPS WL using localization.
What will we do? – continued

• There are many other 3d $N=4$ Chern-Simons-matter theories. [Gaiotto, Witten, 08] [Hosomichi, Lee, Lee, Lee, Park, 08] [Fa-Min Chen, Yong-Shi Wu, 12]

• Are there DT type WL in these theories?

• How about the complete story in $N=3$ case.
Thanks for your time!