Instanton Counting and CFT
Interdisciplinary Center for Theoretical Study, USTC

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Strong-weak duality enable us understanding the non-perturbative aspects in quantum field theories. In general, behind this duality, there is a hidden integrable structure. We propose a new method in triangulating the Hamiltonians which also incorporates the integrability of the models. We have found that our method can apply:
1) explicit construction of the Jack states for the Calogero-Sutherland model in the interacting fermion picture.
2) a complete construction of the AFLT states related to the AGT conjecture on the equivalence of the Nekrasov Instanton Counting (NIC) to the $U(1) \times Vir$ conformal block.
3) generalization to the cases the correspondence between $U(k)$ linear quiver gauge theory and the $U(1) \times WA_{k-1}$

Besides, our construction may facilitate getting new results on the following related research area in mathematical physics: random matrix model and the multiple integrals of the Selberg type.
Based on
1) with Jian-feng Wu, "Calogero-Sutherland model in interacting fermion picture and explicit construction of Jack states", arXiv:1110.6720
2) with Bao Shou, Jian-Feng Wu, "AGT conjecture and AFLT states: a complete construction", arXiv:1107.4784
Quantum field theory ⇒ Perturbations ⇒ Strong coupling ⇒ Duality ⇒ A weakly coupled theory
Introduction

- Quantum field theory ⇒ Perturbations ⇒ Strong coupling ⇒ Duality ⇒ A weakly coupled theory
- Duality ⇒ Integrability ⇒ Triangulating the Hamiltonian ⇒ Similarity transformation ⇒ (Infinitely) many conserved charges
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• Example 1:
  IIB superstrings on the $AdS_5 \times S^5 \Leftrightarrow$ Planar $N = 4$ super Yang Mills theory
  $AdS/CFT$ duality $\Leftrightarrow$ Integrable 1d spin chains
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Duality $\Rightarrow$ A weakly coupled theory

Duality $\Rightarrow$ Integrability $\Rightarrow$ Triagulating the Hamiltonian $\Rightarrow$
Similarity transformation $\Rightarrow$ (Infinitely) many conserved charges

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Example 2:
$N = 2$ super Yang Mills theory coupled to special bifundamental matter
Seiberg-Witten duality $\Leftrightarrow$ Integrable (generalized) Liouville field theory $\Rightarrow$ integrable Calogero-Sutherland model
Spin-chain $\times$ Yangian $\Rightarrow$ Continuum limit $=$
Calogero-Sutherland model
The AGT duality

1994 Seiberg-Witten duality without instanton part.
2002 Nekrasov instanton counting (Omega deformation).

AGT dictionary:
4d SU(2) SCFT full partition function = 2d Liouville correlation function on SW curve
4d 1-loop partition function = 2d DOZZ formula
4d Nekrasov instanton counting = 2d conformal blocks.
Problem: general construction for Liouville conformal blocks.
History: Conformal bootstrap and Teschner’s method, recursively construction, not successful.
Motivation: use 4d theory to solve the construction of Liouville conformal block. Hints:
- $U(2)$ Nekrasov instanton counting: factorization.
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- bi-Jack symmetric function: \[ H_0 J_{\vec{\gamma}} |P\rangle = E_{\vec{\gamma}}(P) J_{\vec{\gamma}} |P\rangle \]
Vir part is a Liouville theory,
\[ [L_n, V_\alpha(z)] = (z^{n+1} \partial_z + (n + 1)hz^n) V_h(z), \quad h = \alpha(Q - \alpha) \]

U(1) part is a free theory but with a peculiar vertex operator, for \( n > 0 \):
\[ [a_n, V_\alpha(z)] = i(Q - \alpha)z^n V_\alpha(z) \]
\[ [a_{-n}, V_\alpha(z)] = -i\alpha z^{-n} V_\alpha(z) \]

- \( L_n = \sum_{k \in \mathbb{Z}} c_k c_{n-k} - inQc_n, \quad c_0 = i\hat{P} \)
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"light-cone" modes of the coupled system is
\[ a_n^\pm = a_n \pm c_n. \]
AFLT Formalism:

\[ \frac{\vec{\mathcal{Y}}'}{\langle P' | V_\alpha | P \rangle} = \prod_{i,j} (Y'_i, -\alpha + P'_i - P_j, Y_j) \]

Here, \( h = \alpha(Q - \alpha), (Y_1, 2P, Y_2) = \prod_{Y_1} (2P + (a_{Y_1} + 1)b^{-1} - l_{Y_2} b) \prod_{Y_2} (2P - a_{Y_2} + (l_{Y_1} + 1)b) \). Here, arm-length \( a_Y(s) = y_i - j, l_Y(s) = y^t_j - i \)

On the 2d CFT side, \( |P\rangle \vec{\mathcal{Y}} \) form an orthogonal basis for the conformal family \( P \) and are called by us as AFLT states.

- Carleson-Okounkov formalism:

\[ \langle J_{Y_1} e^{i(Q + 2P)\mathcal{Y}(-)(1)} e^{i2P\mathcal{Y}(+)(1)} J_{-Y_2} \rangle = (-1)^{Y_1} b^{-1}\mathcal{Y}_1 \mathcal{Y}_2 \mathcal{Y}_1 \mathcal{Y}_2 \]

\( (Y_1, 2P, Y_2) \)
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  \[ H_0 J_{-\vec{Y}} |P\rangle = E_{\vec{Y}}(P)J_{-\vec{Y}} |P\rangle \]

- \( H_0 \) contains terms cubic and quadratic in \( a_n \) and \( c_n \).
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AFLT States

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- We constructed explicitly all the state $|P\rangle_{Y_1,Y_2}$
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AFLT-AGT state for generic Liouville conformal block.

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and Reflection $\rightarrow$ insertions of screening charges.
from AFLT’s paper, arXiv:hep-th/1012.1312v3, ”It would be very naive to expect to find analytical expressions for all the states $|P\rangle \vec{Y}$ in a closed form. We have constructed all the states $|P\rangle \vec{Y}$ up to level 6. They are all polynomials in $P$ of degree $|\vec{Y}|$. Some of the coefficients of these polynomials have a very suggestive form, but we were unable to find a closed form expression for all of them.”
Construction of AFLT States

Reflection symmetry in NIC:
\[ Y_1, Y_2 \langle P | \leftrightarrow Y_2, Y_1 \langle -P | . \text{ or } | P \rangle_{Y_1, Y_2} \leftrightarrow | -P \rangle_{Y_2, Y_1}. \]

On the 2d CFT side, screening charges incertions with 
\[ 2p + 2nb = 0 \]
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- AFLT States: \( \mid P \rangle_{\tilde{Y}} = RJ^+_{\gamma_1} J^-_{\gamma_2} \mid P \rangle \Omega_{\tilde{Y}}(P) \)

- R Matrix Triangular: \( R = 1 + \cdots = 1 + \tilde{R} \)
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- **R Matrix Triangular:** \[ R = 1 + \cdots = 1 + \tilde{R} \]
- **Ω Normalization:** \[ \Omega_{\vec{Y}} = (-1)^{|Y_1|b} |Y_1|^+ |Y_2| (Y_1, 2P, Y_2) \]
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- R Matrix Triangular: $R = 1 + \cdots = 1 + \tilde{R}$
- $\Omega$ Normalization: $\Omega_{\tilde{Y}} = (-1)^{|Y_1| b |Y_1| + |Y_2|} (Y_1, 2P, Y_2)$
- $R = \frac{1}{1 - \frac{1}{E_{\tilde{Y}}(P) - H_0} H_I}$
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- \( \Omega \) Normalization: \[ \Omega_{\gamma} = (-1)^{|Y_1|b|Y_1|+|Y_2|}(Y_1, 2P, Y_2) \]
- \[ R = \frac{1}{1 - \frac{1}{E_{\gamma}(P) - H_0} H_1} \]
- \[ H = H_0 + H_1, H | P \rangle_{\gamma} = E_{\gamma}(P) | P \rangle_{\gamma}, \]

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With the above choice, there is still a freedom in choosing the triangular term $H_I$. Since we are in fact working with $U(1) \times Vir$ system, we have to impose $[H, S] = 0$ with $S$ the screening charge. This fix $H_I = \sum_{n=1}^{\infty} 2Qn a_n^+ a_n^-$

$W^\infty$ symmetry: $H^{(n)} |P\rangle_\bar{\gamma} = E^{(n)}_\bar{\gamma} (P) |P\rangle_\bar{\gamma}$, $[H^{(n)}, H^{(m)}] = 0$

They are related to the the so called second Hamiltonian structure of KdV type
$H$ can be decomposed into $H = H^0 + H^I$, and $H^0|E_0\rangle_0 = E_0|E_0\rangle_0$ is known.

Then the energy eigenequation $H|E\rangle = E|E\rangle$ can be solved with the following solution:
Triangulating the Hamiltonian

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\[
E = E_0 + E_1, \quad H^d = H^0 + H^\parallel \\
|E\rangle = \frac{1}{1 - \frac{1}{E - H^d} H^\perp} |E_0\rangle_0
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\[
|E⟩ = \frac{1}{1 - \frac{1}{E - H^d}H^\perp}|E_0⟩_0
\]

\[
|E⟩ = \sum_{n=0}^{∞}(\frac{1}{E - H^d}H^\perp)^n|E_0⟩_0
\]
Proof of the eigenstate

To see that this construction really solves the eigenequation problem, we first rewrite $H$ as

$$H = E + H^d + H^\perp - E = E + (H^d - E)(1 + \frac{1}{H^d - E}H^\perp).$$
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Then we have

$$H|E\rangle = E|E\rangle + (H^d - E)|E_0\rangle_0,$$

$$= E|E\rangle$$

Hence, we conclude that $|E\rangle$ is an eigenstate of $H$ with eigenvalue $E = E_0 + E_1$. 

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Some clarification

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- Within each multiplet with the leading term a father state, each family member has a distinct $H^d$ eigenvalue for generic values of the parameters on which $H^d$ may depend.

- the eigenstate of $H$, constructed this way, is actually the common eigenstate for all the conserved charges which commute with $H$. 

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The similarity transformation $S$

\[ S = T \exp \left( \int_{-\infty}^{0} H^\perp(t) dt \right), \]

\[ H^\perp(t) = \exp(-tH^d)H^\perp \exp(tH^d). \]

$T$ time ordering, larger $t$ to the left, $H^\perp(-\infty) = 0$

\[ S|E^{(0)}\rangle_0 = (1 - (E - H^d)^{-1}H^\perp)^{-1}|E^{(0)}\rangle_0 = |E\rangle. \]

$S$ adiabatically turns on $H^\perp$ from time $-\infty$ to time $0$.

\[ S^{-1} = T \exp(-\int_{0}^{\infty} H^\perp(-t) dt). \]

The orthogonality

\[ 0\langle E^{(0)}|S^{-1} = 0\langle E^{(0)}|(1 - H^\perp(E - H^d)^{-1})^{-1} = \langle E|. \]

$H = H^d + H^\perp = SH^d S^{-1}$. 

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Calogero-Sutherland model as an example

The CS model for studying $N$ interacting particles on a circle of circumference $L$, $\hbar^2/m = 1$, $L = \pi$.

$$H_{CS} = -\sum_{i=1}^{N} \frac{1}{2} \partial_{x_i}^2 + \sum_{i<j} \frac{\beta(\beta - 1)}{\sin^2(x_{ij})}. $$

eigenfunctions $\Psi_0(\{x_i\}) = \prod_{i<j} \sin^\beta(x_{ij})$, $J_{\lambda}(\{z_i\})$ is the Jack symmetric polynomial with $z_j = \exp(2ix_j)$.

$\Psi_{\lambda}(\{x_i\}) = \Psi_0(\{x_i\}) J_{\lambda}^{1/\beta}(\{z_i\})$. eigenvalue $2E_{\lambda}$,

$$E_{\lambda} = \sum P_i^2, \quad P_i = \lambda_i + \beta((N+1)/2 - i).$$

triangular in symmetric monomial basis $m_\mu$,

$$J_{\lambda}^{1/\beta}(\{z^n_i\}) = \sum_{\mu<\lambda} (v^{1/\beta})_{\lambda}^{\mu} m_{\mu}$$

$$\mu < \lambda \Rightarrow \sum_{i=1}^{j} \mu_i < \sum_{i=1}^{j} \lambda_i, \text{ for } j = 1, 2, \cdots .$$
Hamiltonian for the collective motion of the CS model:

\[
H_{CS} = k \sum_{n,m>0} (a_{-n}a_{-m}a_{n+m} + a_{-n}a_n a_m) + \sum_{n>0} (N + (1 - \beta) n) a_{-n}a_n ,
\]

\[
\beta = k^2, \ [a_n, a_m] = n\delta_{n+m,0} .
\]

\[
V_k(z_i) = \exp(k \sum_{n>0} a_{-n}z_i^n / n) \exp(-k \sum_{n>0} a_n z_i^{-n} / n) e^{kq \sum_{n>0} z_i^n} \left| k_{in} \right> ,
\]

\[
\Psi_\lambda(\{x_i\}) = \langle k_f | J_\lambda \prod_{i=1}^N V_k(z_i) | k_{in} \rangle.
\]

\[
a_0 | k_{in} \rangle = k_{in} | k_{in} \rangle, \ k_{in} = -(N - 1)k/2, \ k_f = (N + 1)k/2.
\]

\[
\langle 0 | J_\lambda H = \langle 0 | J_\lambda (E_\lambda - E_0).
\]

\[
H_{CS} \Psi_\lambda(\{x_i\}) = \langle k_f | J_\lambda (H + E_0) 2 \prod_{i=1}^N V_k(z_i) | k_{in} \rangle = 2E_\lambda \Psi_\lambda(\{x_i\}).
\]

\[
J_\lambda's \ in \ the \ power \ sum \ basis, \ J_\lambda \equiv J_\lambda^{1/\beta}(\{a_n/k\})
\]
Fermionic Schur states

$H_{CS}$, in bosonic form, is not triangular. However, we know that for $\beta = 1$, the Jack states reduce to the Schur states, a nonrelativistic free fermion theory. $H_{CS}$ is an interacting fermion theory with perturbation parameter $\beta - 1$.

- the fermionic picture, $a_n = \sum_{r \in \mathbb{Z} + 1/2} : b_{n-r} c_r :$. with
  \[ \{b_r, c_s\} = \delta_{r+s,0} \quad r, s \in \mathbb{Z} + 1/2 \] and $b_r |0\rangle = c_r |0\rangle = 0$, $r > 0$. 

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- The Schur states are labeled by the Maya diagram, and can be translated into the Young diagram.
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- Each Schur state is created by a fermion monomial with equal number of $b_{-r}$’s and $c_{-s}$’s actiong on the vacuum state $|0\rangle$.

$\vec{r}, \vec{s} = \prod_i b_{-r_i} c_{-s_i} |0\rangle$. 

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- The correspondence between the Schur function and the fermionic state as following:

$$|\lambda\rangle \equiv s_\lambda |0\rangle = (-1)^{\sum_{i=1}^{d(\lambda)} (1/2-s_i)} \prod_{i \in \lambda, \lambda_i \neq 0} b_{-r_i} c_{-s_i} |0\rangle.$$

with $r_i = \lambda_i - i > 0$, and $s_i = \lambda_i^t - i > 0$. 
The Schur states are the eigenstates of the CS Hamiltonian at $\beta = 1$:

$$H^0 = \sum_{r>0} [r^2 + \frac{3}{4}(b_r c_r - c_r b_r)],$$

with the eigenvalue

$$E^0_\lambda = \sum_{i=1}^{d(\lambda)} (r^2_i - s_i^2) = \sum_{i=1}^{\lambda^t_1} \lambda^2_i - \sum_{i=1}^{\lambda_1} (\lambda^t_i)^2$$

Notice that the fermionic description for Schur function is just a very simple monomial. This suggests one may rewrite the Hamiltonian as a fermionic formalism, and then find a fermionic representation for the Jack state. The Jack state, should represent itself as a module generated by the Schur function.
the fermionic formalism for the CS Hamiltonian

\[ H_{CS} = H_0 + H^\parallel + H^\perp \]
the fermionic formalism for the CS Hamiltonian

\[ H_{CS} = H_0 + H^\parallel + H^\perp \]

\[
H^\parallel = \sum_{r>0} \frac{1}{3} (1 - \beta)(r - \frac{1}{2}) b_{-r} c_r \\
+ \sum_{r>0} (1 - \beta)(r - \frac{1}{2})(r + \frac{1}{6}) c_{-r} b_r \\
+ \sum_{r+s>0} \frac{2}{3} (2r + s)(1 - \beta) : b_{-s} c_{-r} b_r c_s : 
\]

\[
H^\perp = (1 - \beta) \times \\
\sum_{r+s>0} (2r + \frac{2}{3}(s + l)) : b_k c_l b_r c_s : . 
\]
\[ H^\perp \text{ strictly triangular, 5 subprocesses,} \]

\[
\frac{1}{2(1-\beta)} H^\perp = \sum_{n=1, r>s>0}^{n=s-1/2} (s-r)c_{-r-n}c_{-s+n} b_s b_r \\
+ \sum_{n=1, r>s>0}^{n=[(r-s-1)/2]} (s-r+2n)b_{-r+n}b_{-s-n}c_s c_r \\
+ \sum_{n=1, r,s>0}^{n=s-1/2} (r+s-n)b_{-s+n}c_{-r-n} b_r c_s \\
+ \sum_{r>l>0, s>0} (l-r)c_{-l-s-r} b_l b_r c_s \\
+ \sum_{l>r>0, s>0} (l-r)b_{-l}c_{-s} b_{-r} c_{r+s+1},
\]

here \([x]\) stands for the integer part of the number \(x\).
Bosons or fermions

- $H^\perp$ is actually triangular, hidden behind its expression. By a careful analysis, one can recognize it explicitly.
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- \( H^\perp \) is actually triangular, hidden behind its expression. By a careful analysis, one can recognize it explicitly.

- \( H^\parallel \) shifts the eigenvalue of the Schur state to

\[
E^{1/\beta}_\lambda = \sum_{i=1}^{\lambda_1^t} \{ \lambda_i^2 - \beta(2i - 1)\lambda_i \},
\]

which is the eigen-energy of the Jack state. \( H^\perp \), however, only changes the fermionic monomial (Schur state) to fermionic polynomial and does not change the eigenvalue.
• $H^\perp$ is actually triangular, hidden behind its expression. By a careful analysis, one can recognize it explicitly.

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$$E_\lambda^{1/\beta} = \sum_{i=1}^{\lambda_i^t} \{ \lambda_i^2 - \beta(2i - 1)\lambda_i \},$$

which is the eigen-energy of the Jack state. $H^\perp$, however, only changes the fermionic monomial (Schur state) to fermionic polynomial and does not change the eigenvalue.

• If this method is valid, then the final state should be the Jack state and one will get the fermionic representation for the Jack symmetric function.
Examples: level 3 cases

We have checked this fermionic formalism for Jack states up to level 4, all of them are in perfect agreement with the corresponding bosonic formalism. Here we show the level 3 results,

\[
J^{1/\beta}_{\{1,1,1\}} = 6s_{\{1,1,1\}},
\]

\[
J^{1/\beta}_{\{2,1\}} = \frac{2\beta + 1}{\beta} s_{\{2,1\}} + \frac{2(\beta - 1)}{\beta} s_{\{1,1,1\}},
\]

\[
J^{1/\beta}_{\{3\}} = \frac{(\beta + 2)(\beta + 1)}{\beta^2} s_{\{3\}}
\]

\[
+ \frac{2(\beta - 1)(\beta + 1)}{\beta^2} s_{\{2,1\}} + \frac{(\beta - 1)(\beta - 2)}{\beta^2} s_{\{1,1,1\}}.
\]

We can expand the Schur polynomial in the basis of power-sum polynomials and the transition coefficient is proportional to the character for the related representation evaluated in the conjugacy class of symmetric group.
Quotation from Michel Lassalle and Michael Schlosser, ”Inversion of the Pieri formula for Macdonald polynomials”, arXiv:math/0402127, ”Fifty years ago, Hua [3] introduced a new family of polynomials defined on the space of complex symmetric matrices, and set the problem of finding their explicit analytic expansions in terms of Schur functions [3, p. 132, Eq. (6.2.5)].” Jacks’ corresponds to $\beta$-deformed matrix model. So we solved the similar problem for general $\beta$ within the framework of the quantum field theory.
Martin HALLNAS, ”An Explicit Formula for Symmetric Polynomials Related to the Eigenfunctions of Calogero Sutherland Models”, SIGMA 3 (2007), 037, ”There remains then to actually compute the matrix elements $c_{\lambda \mu}$, i.e., to collect all terms in (8) corresponding to the same monomial $m_{\mu}$. It seems however that this problem does not have a simple solution, which in turn implies that the reduced eigenfunctions of the Calogero model do not have a simple series representation in terms of monomial symmetric polynomials. The situation is similar for the other models discussed above (see e.g. [8]), and as far as we know also for other simple bases of the space of symmetric polynomials, such as elementary, complete homogeneous and power sum symmetric polynomials;”
integrability and deformed $W^\infty$ algebra

put $N$ vertex operators $V_k(z_i)$’s acting successively on the $|k_{in} - k/2\rangle$ vacuum starting from $V_k(z_N)$. the resulting modes labeled by Young tableau (with maximal $N$ rows),

$V_{(1-N)\beta/2-\lambda_1} \cdots V_{(N-1)\beta/2-\lambda_N} |k_{in} - k/2\rangle$.

i-th mode carries the momentum $((N+1)/2 - i)\beta + \lambda_i$

CS energy = summing over each momentum square.

$\beta = 1$, free fermion theory (NS sector $\Rightarrow N \in \text{even}$).

define a momentum operator for $b_{-r}$,

$P_r^{(0)} = \times \times rb_{-r}c_r \times \times$.

normal ordering $\times \times \times \times$ respect to the “empty” vacuum $|k_{in} - 1/2\rangle$ at $\beta = 1$, $b_r|k_{in} - 1/2\rangle = 0$, $r > N/2$.

$$\times b_r c_s \times = \begin{cases} b_r c_s, & \text{if } r < N/2; \\ -c_s b_r, & \text{if } r > N/2. \end{cases}$$
\( \beta \neq 1 \) CS model, we stay in the fermionic picture, define the shifted momentum operator \( P^d_r \equiv P^{(0)}_r + P^\parallel_r = \sum_{r} b_{-r} c_r \left( r + (\beta - 1)((N + 1)/2 - \sum_{s \geq r} b_s c_s) \right) \).

The ground state is specified by the null Young tableau, and the Fermi sea is filled up to momentum \((N - 1)/2\). Filled Fermi sea \( b_r |f\rangle = 0 \) for \( r > -N/2 \). The following normal ordering

\[
: b_r c_s : = \begin{cases} 
  b_r c_s, & \text{if } r < -N/2; \\
  -c_s b_r, & \text{if } r > -N/2. 
\end{cases}
\]

Now define \( H^d = \sum_r (P^d_r)^2 \). If the i-th electron’s momentum is moved up exactly by \( \lambda_i \) amount, then \( H^d \) acts on this system will produce the exact CS spectrum. \( S(k) \) act on this fermion monomial state will produce the exact Jack state. Since \( [P^d_r, P^d_s] = 0 \), the conserved charges can now be constructed,

\[
W^n = S(k) \sum_r (P^d_r)^n S^{-1}(k) \Rightarrow [W^n, W^m] = 0, \ n, m > 0.
\]
we do not use the other commonly used method in integrable systems, like Bethe ansatz, Lax pair, Quantum group, Dunkl operator, Sekiguchi operator etc. There must be a deep connection between our method and the others.

Generalization:

1) Jack symmetric function $\rightarrow$ Macdonald symmetric function.

4) one external leg $\rightarrow$ full pants diagram

Work in progress:

1) Vir $\rightarrow$ $WA_{k-1}$ algebra

2) $c \geq 25 \rightarrow c < 25$

3) Explicit evaluation of the Jack symmetric function in the bosonic formalism (power sum basis),

$$J^\alpha_Y = (a_1/k)^n + f_1(\alpha)(a_1/k)^{n-2}a_2/k + \cdots .$$
Thanks