Higher derivative gravity and brane world

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1. Introduction and motivation
1. Introduction and motivation (cont.)

The prototype idea of brane world was proposed during early 1980s and made great progresses after the Arkani-Hamed-Dimopoulos-Dvali (ADD) model and Randall-Sundrum (RS) models proposed in the late 1990s.

It suggests that the SM particles are trapped in a 4D hypersurface (called brane) embedded in a higher dimensional space-time (called bulk). It provides us new perspectives to solve the gauge hierarchy problem and the cosmological constant problem.
The brane configuration is determined by

- Gravity theory;
- Distribution of the matter;
- The ways of gravity-matter coupling.

According to the thickness of the brane, there are two categories

- Thin branes: $\delta$-function source and junction condition;
- Thick branes: smooth source.
Motivated by the works of chiral topologically massive gravity with a negative cosmological constant in 3D, H. Lü and C. N. Pope constructed critical gravities (quadratic-curvature actions with a cosmological constant) in 4D (Phys. Rev. Lett. 106, 181302(2011)).

At the critical point, these theories possess an anti-de Sitter (AdS) vacuum, for which there is only a massless tensor. It was also shown that at the critical point the theory admits additional modes, namely, the so-called logarithmic modes, which arise as limits of the massive spin-2 modes of the noncritical theory.
In this report, we reconstruct a brane model in the simplest higher derivative gravity but at the critical point.

The action for a thin brane is

\begin{align}
S &= S_g + S_b, \\
S_g &= \frac{1}{2\kappa^2} \int_M \left[ R - 3\Lambda_0 + \alpha R^2 + \beta R_{MN} R^{MN} + \gamma \mathcal{L}_{GB} \right], \\
S_b &= \int_{\partial M} (-V_0),
\end{align}

where $\kappa^2 = 8\pi G_5$, $\int_M \equiv \int_M d^5x \sqrt{-g}$, $\int_{\partial M} \equiv \int d^4x \sqrt{-q}$, $\mathcal{L}_{GB} = R^2 - 4R_{MN} R^{MN} + R_{MNPQ} R^{MNPQ}$ is the Gauss-Bonnet term, $q_{\mu\nu}$ is the induced metric on the brane, and $V_0$ is the brane tension.
The equations of motion are given by

\[ G_{MN} + \alpha E^{(1)}_{MN} + \beta E^{(2)}_{MN} - \frac{1}{2} \gamma H_{MN} = \kappa^2 T_{MN}, \]  

where \( T_{MN} = -V_0 \delta^\mu_M \delta^\nu_N g_{\mu\nu} \delta(y), \) and

\[ G_{MN} = R_{MN} - \frac{1}{2} R g_{MN} + \frac{3}{2} \Lambda_0 g_{MN}, \]
\[ E^{(1)}_{MN} = 2R \left( R_{MN} - \frac{1}{4} R g_{MN} \right) + 2g_{MN} \Box R - 2 \nabla_M \nabla_N R, \]
\[ E^{(2)}_{MN} = 2R^{PQ} \left( R_{MPNQ} - \frac{1}{4} R_{PQ} g_{MN} \right) + \Box \left( R_{MN} + \frac{1}{2} R g_{MN} \right) - \nabla_M \nabla_N R, \]
\[ H_{MN} = g_{MN} \mathcal{L}_G - 4R R_{MN} + 8R_{MP} R^P_N + 8R_{MANB} R^{AB} - 4R_{MABC} R^A_N R^{BC}. \]
1. Introduction and motivation (cont.)

The line element describing a static flat brane is

\[ ds^2 = g_{MN} dx^M dx^N = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad (4) \]

where \( e^{2A} \) is the warp factor.

It is very difficult to find solutions for arbitrary \( \alpha, \beta, \) and \( \gamma \) for the fourth-order differential equations.

However, at the critical point \( 16\alpha + 5\beta = 0 \), the equations of motion (EOMs) in the bulk are reduced to the second-order ones:

\[ \Lambda_0 + 4A'^2 + \zeta A'^4 = 0, \quad (5a) \]
\[ (2 + \zeta A'^2) A'' = 0. \quad (5b) \]
If we apply pill-box integral to EOMs (5), the junction condition reads

\[ \int_{0^-}^{0^+} dy \frac{3}{2} (2 + \zeta A'^2) A'' = \left( 3A' + \frac{\zeta}{2} A'^3 \right) \bigg|_{0^-}^{0^+} = -\kappa^2 V_0, \quad (6) \]

where the prime denotes the derivative with respect to \( y \), and

\[ \zeta \equiv 3\beta - 8\gamma. \quad (7) \]
2. Boundary term and junction condition
However, the junction condition given by pill-box method is not full satisfaction. To have a well-posed variation, we will adopt Gibbons-Hawking method.

**Basic idea**

- The thin brane divides the whole space-time $M$ into two submanifolds and should be interpreted as the boundary $\partial M$ of the two submanifolds;
- $n^Q$ is the unit vector normal to the boundary $\partial M$ and **outward pointing**. $q^{MN} = g^{MN} - n^M n^N$ is the induced metric on the brane. $K_{MN} = \mathcal{L}_{\bar{n}} q_{MN}/2$ is the extrinsic curvature, and $K = g^{MN} K_{MN}$ is the trace of the extrinsic curvature;
- The directions of $n^Q$ on both sides of $\partial M$ are opposite. If we fix the vector $n^Q$, the final results can be written as $[\bullet]_\pm$, where $[F]_\pm := F(0+) - F(0-)$. 

2. Boundary term and junction condition (cont.)

To proceed, we rewrite the action. The squared Riemann tensor can be rewritten

\[ \alpha R^2 + \beta R_{MN} R^{MN} + \gamma \mathcal{L}_{\text{GB}} = \frac{3\beta}{8} C^2 - \frac{\zeta}{8} \mathcal{L}_{\text{GB}} + \frac{16\alpha + 5\beta}{16} R^2. \] (8)

Here \( C^2 := C^{MNPQ} C_{MNPQ} \) is the square of the 5D Weyl tensor.

It is obvious that \( 16\alpha + 5\beta = 0 \) and \( \zeta = 8\gamma - 3\beta = 0 \) are special. Since the Weyl tensor vanishes for flat brane, when \( 16\alpha + 5\beta = 0 \) is satisfied, the solutions of the EOMs as well as the junction condition are the same as the Einstein-Gauss-Bonnet (EGB) gravity. Applying EGB junction condition to our case, it does give eq.(6).
For a general warped geometry with 
\[ ds^2 = e^{2A(y)} \hat{g}_{\mu\nu}(x) dx^\mu dx^\nu + dy^2 = e^{2A} (\hat{g}_{\mu\nu} dx^\mu dx^\nu + dz^2), \]
we expect that the junction condition is also of first order in the critical gravity, because \( C^{MPNQ} \) is continuous.

However, in this case, the solutions of the EGB gravity do not satisfy the EOMs of the critical gravity.

We can also prove this statement from the full variational principle. This means that we should start from the action of the general case instead of the warped geometry.
Explicitly, we have

\[ \delta \int_M C^2 = \int_M \left[ 2 C_M^{PQR} C_{NPQR} - \frac{1}{2} g_{MN} C^2 \right. \]
\[ + \left. \frac{8}{3} R^{PQ} C_{MPNQ} - 4 C_{(M \; N) ; PQ} \right] \delta g^{MN} \]
\[ + \frac{4}{3} \int_{\partial\Omega} \left[ \left( C^{MPNQ}_{\quad ; P} n_Q \delta g_{MN} \right) ; P \right. \]
\[ - \left. \left( \left( C^{MPNQ}_{\quad ; P} n_Q \right) ; P + C^{MPNQ}_{\quad ; Q} n_P \right) \delta g_{MN} \right]. \quad (9) \]

The bulk term gives contribution to the EOMs, and the boundary term (and the corresponding generalized Gibbons-Hawking term) will give contribution to the corresponding junction condition.
In order to do this, we introduce an auxiliary field $\varphi^{MNPQ}$, which has the same symmetry as the Weyl tensor and is also totally traceless.

So $C^2$ is replaced by $2\varphi^{MNPQ}C_{MNPQ} - \varphi^{MNPQ}\varphi_{MNPQ}$. Its EOM is $\varphi^{MNPQ} = C^{MNPQ}$. Then we replace $C^{MNPQ}$ by the new field $\varphi^{MNPQ}$ in the action.

We should notice that the fields $\delta\phi_i, L_{\bar{n}}\delta\phi_i$ on the boundary are independent, where $\phi$ is any field. The following identities are vital

\begin{align}
\delta n_M &= -\frac{1}{2} n_M n_P n_Q \delta g^{PQ}, \tag{10} \\
X^{M}_{,M} &= D_M(q^M_N X^N) + K n_N X^N + L_{\bar{n}}(n_N X^N). \tag{11}
\end{align}
With the help of the identity (11), after integrating out the pure divergence \( D_M(q^M_N X^N) \), we get

\[
\delta S_{C^2} = \frac{3\beta}{4\kappa^2} \int_{\partial M} \left[ 2\varphi^{MN} \delta \bar{K}_{MN} \\
+ (W^{MN} - \varphi^{PQ} K_{PQ} n^M n^N) \delta g_{MN} \right],
\]

(12)

where

\[
\bar{K}_{MN} = K_{MN} - \frac{1}{4} q_{MN} K,
\]

(13)

\[
W^{MN} = \frac{3}{2} K \varphi^{MN} + \mathcal{L} \bar{n} \varphi^{MN} - 2\varphi^P_{;P} (M n^N)
- (\varphi^{MPNQ} n_Q) ;P - \varphi^{MPNQ} ;Q n_P,
\]

(14)

\[
\varphi^{MN} : = \varphi^{MPNQ} n_Q n_P.
\]

(15)

All of these tensors are perpendicular to normal and traceless.
Now we can introduce the corresponding Gibbons-Hawking surface term for the $C^2$ term

$$S_{\text{CGH}} = -\frac{3\beta}{2\kappa^2} \int_{\partial M} \varphi^{MN} \bar{K}_{MN}. \quad (16)$$

So we have

$$\delta(S_{C^2} + S_{\text{CGH}}) = \frac{3\beta}{4\kappa^2} \int_{\partial M} \left\{ 2[\bar{K}_{MN}]_\pm \delta \varphi^{MN} 
- [\varphi^{PQ} K_{PQ}]_\pm n^M n^N \delta g_{MN} 
+ [W^{MN}]_\pm \delta g_{MN} \right\}. \quad (17)$$
The junction conditions are

\[
\begin{align*}
[\bar{K}_{MN}]_\pm &= 0, \\
[\varphi^{PQ} K_{PQ}]_\pm &= \bar{K}_{PQ}[\varphi^{PQ}]_\pm = 0, \\
-\frac{3\beta}{2} [W^{MN}]_\pm + [E^{(GB)MN}]_\pm &= -\kappa^2 T_{(brane)}^{MN}.
\end{align*}
\]

To avoid \(\delta\)-function in the junction conditions, we need stronger condition \([\varphi^{MN}]_\pm = 0\) (like the constraint \([g_{MN}]_\pm = 0\)). Then it is easy to prove that the results do not depend on the choice of any basic field.
3. Perturbation
In order to study the effective four-dimensional gravity on the branes, we need to consider the perturbations of the background metric:

$$ds^2 = e^{2A(z)}(\eta_{MN} + \tilde{h}_{MN})d\xi^M d\xi^N. \quad (21)$$

The perturbation $\tilde{h}_{MN}$ can be decomposed into transverse traceless (TT) and nontransverse traceless (NT) parts:

$$\tilde{h}_{MN} = \tilde{h}_{MN}^{TT} + \tilde{h}_{MN}^{NT}, \quad (22)$$

$$\tilde{h}_{MN}^{NT} = \partial_{(M} f_{N)}(x, z) + g(x, z)\eta_{MN}, \quad (23)$$

for some functions $f_N(x, z)$ and $g(x, z)$. 
3. Perturbation (cont.)

The Weyl tensor $C^M_{NPQ}$ is a conformal invariance, so we can calculate its perturbations in flat spacetime.

\begin{equation}
\delta F \rightarrow \delta F + \mathcal{L}_\xi F, \quad x^M \rightarrow x^M - \xi^M.
\end{equation}

Since the Weyl tensor in the brane background vanishes, the tensor $\delta C^M_{NPQ}$ is gauge invariant. Note the NT component can be canceled by the gauge and conformal transformations, $\delta C^M_{NPQ}(\bar{h}_{RS}) = \delta C^M_{NPQ}(\bar{h}^{TT}_{RS})$ for flat spacetime.

If we choose the axial gauge $\bar{h}_{5M} = 0$, then the transverse traceless TT condition means

$\eta^{\mu\nu} \partial_\mu \bar{h}^{TT}_{\nu\rho} = 0 = \eta^{\mu\nu} \bar{h}^{TT}_{\mu\nu}$. The NT perturbation equations are the same as that of the EGB gravity.
Since the NT and TT components of the fluctuations are decoupled, and the NT components do not contribute to the $C^2$ part, the NT (scalar) perturbation equations are the same as that of the EGB gravity.

So, it also can be shown that the scalar perturbations are stable for our brane models, and the scalar zero modes are not localized on the brane.

This is very important for a brane model, because localized scalar zero modes would lead to a “fifth force” never observed and is unacceptable in the effective four-dimensional theory.
4. Summary
For critical gravity, equations of motion become second-order.

In order to obtain the covariant junction condition, irreducible components are very important. $C_{MNPQ}$ is an irreducible component of the Riemann curvature, which results that the corresponding junction condition just contains the tensor $\bar{K}_{MN}$. 

The scalar perturbations are stable for our brane models, and the scalar zero mode is not localized on the brane. Critical gravity also has additional spin-2 modes, which are described by the TT modes.

Open issues: ghost, unitary, stability of TT perturbation...
Thank you for your listening!

chenfw10@lzu.edu.cn