Loop Quantum Brans-Dicke Gravity and Its Cosmological Application

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Based on the joint work with Michal Artymowski and Xiangdong Zhang
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The Basic Ideas of LQG

- The application of perturbative quantization to GR fails due to its nonrenormalizability.

\[ g_{ab} = \eta_{ab} + h_{ab}. \]

The separation of the gravitational field from background spacetime is in strident contradiction with the very lesson of GR.
The Basic Ideas of LQG

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- LQG inherits the basic idea of Einstein that gravity is fundamentally spacetime geometry.
  Hence the theory of quantum gravity is a quantum theory of spacetime geometry with diffeomorphism invariance.
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- The choice of the algebra of field functions to be quantized: Not the positive and negative components of the field modes as in conventional QFT; but the holonomies of the gravitational connection and the electric flux.
Success of LQG and Its Scope

- It is remarkable that, as a non-renormalizable theory, GR can be non-perturbatively quantized by the loop quantization procedure. What is the applicable scope of loop quantum gravity?
  - LQG can be extended to $f(R)$ theories of gravity [Zhang, YM, 2011].
  - LQG is applicable to GR in arbitrary dimensions [Bodendorfer, Thiemann, Thurn, 2011].
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- To explain the accelerated expansion of the universe, as well as dark matter, from fundamental physics is now a great challenge.

- A large variety of models of \( f(\mathcal{R}) \) modified gravity have been proposed to account for the dark energy and the dark matter problems. [Sotiriou and Faraoni 2010]
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- A large variety of models of $f(R)$ modified gravity have been proposed to account for the dark energy and the dark matter problems. [Sotiriou and Faraoni 2010]
- Historically, Einstein’s GR is the simplest relativistic theory of gravity with correct Newtonian limit. It is worth pursuing all alternatives, which provide a high chance to new physics.
Gravity as Geometry

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- If some modified gravity theory becomes fundamental rather than GR, one has to consider its quantization as well.
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- Besides GR, Brans-Dicke theories belong to metric theories of gravity.
- For metric theories, gravity is still geometry with diffeomorphism invariance as in GR.
  The differences between them are just reflected in dynamical equations and additional variables.
  Hence, a background-independent and non-perturbative quantization for metric theories of gravity is preferable.
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- If some modified gravity theory becomes fundamental rather than GR, one has to consider its quantization as well.
- Besides GR, Brans-Dicke theories belong to metric theories of gravity.
- For metric theories, gravity is still geometry with diffeomorphism invariance as in GR. The differences between them are just reflected in dynamical equations and additional variables. Hence, a background-independent and non-perturbative quantization for metric theories of gravity is preferable.
- Since general Brans-Dicke theories (BDT) include $f(R)$ theories as special case and have received increased attention due to motivations coming from cosmology and astrophysics, we will take them as examples to carry out the extension of LQG to metric theories.
Connection Dynamics of GR

- The Hamiltonian formulation of GR can be cast into the connection dynamical formalism, where the configuration and conjugate momentum are defined respectively by [Ashtekar 1986; Barbero, 1994]:

\[
\begin{align*}
A^i_a &:= \Gamma^i_a + \gamma K^i_a, \\
\tilde{P}^a_i &:= \frac{1}{2\kappa\gamma} \tilde{\eta}^{abc} \epsilon_{ijk} e^j_b e^k_c = \frac{1}{\kappa\gamma} \sqrt{\det q} e^a_i.
\end{align*}
\]

Here \( \Gamma^i_a \) is the \( SU(2) \)-spin connection on a 3d spatial manifold \( \Sigma \), \( K^i_a \) is the extrinsic curvature of \( \Sigma \), and \( \gamma \) is the Barbero-Immirzi parameter.
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- It turns out that the configuration variable \(A^i_a\) performs as a \(su(2)\)-valued connection on \(\Sigma\).

- The Hamiltonian density \(H_{tot}\) is a linear combination of first-class constraints: 
  \[
  H_{tot} = \Lambda^i G_i + N^a V_a + NS.
  \]
Loop Quantization of GR

- The kinematical Hilbert space: $\mathcal{H}_{kin} = L^2(\mathcal{A}, d\mu^0)$.
- Quantum connections on a finite graph $\alpha$: $\mathcal{A}_\alpha \in \mathcal{A}_\alpha$ can be identified with the holonomies of $A_\alpha$ along the edges of $\alpha$.
- The family of induced Haar measures $\mu^0_\alpha$ on $\mathcal{A}_\alpha$ defines a regular Borel measure $\mu^0$ on $\mathcal{A}$ via the projective techniques.
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- The family of induced Haar measures $\mu^0_\alpha$ on $\mathcal{A}_\alpha$ defines a regular Borel measure $\mu^0$ on $\mathcal{A}$ via the projective techniques.
- Uniqueness Theorem [LOST, 2005]: There is a unique gauge and diffeomorphism invariant representation of the holonomy-flux $\star$-algebra, given by $\mu^0$. 

![Diagram](image.png)
Geometric operators

- Area operator [Rovelli and Smolin, 1995; Ashtekar and Lewandowski, 1997]

  Given a closed 2-surface or a surface $S$ with boundary, its area can be well defined as a self-adjoint operator $\hat{A}_S$ on $\mathcal{H}_{kin}$:

  $$\hat{A}_S \psi_\alpha = 4\pi \gamma \ell_p^2 \sum_{v \in V(\alpha \cap S)} \sqrt{(\hat{j}_{i(u)}^{(S,v)} - \hat{j}_{i(d)}^{(S,v)})(\hat{j}_{j(u)}^{(S,v)} - \hat{j}_{j(d)}^{(S,v)})}\delta_{ij} \psi_\alpha.$$
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One can find some finite linear combinations of spin network basis in $\mathcal{H}_{\text{kin}}$ which diagonalize $\hat{A}_S$ with eigenvalues:

$$a_S = 4\pi \gamma \ell_p^2 \sum_{l} \sqrt{2j(u)(j(u) + 1) + 2j(d)(j(d) + 1) - j(u+d)(j(u+d) + 1)},$$

where $j(u), j(d)$ and $j(u+d)$ are arbitrary half-integers subject to the condition: $j(u+d) \in \{|j(u) - j(d)|, |j(u) - j(d)| + 1, ..., j(u) + j(d)|\}$. 
Geometric operators

- Thus the spectrum of the area operator is fundamentally pure discrete, while its continuum approximation becomes excellent for large eigenvalues.

- Volume operator [Ashtekar and Lewandowski, 1995, 1997; Rovelli and Smolin, 1995]
  The volume of a compact region $R$ can also be well defined as a self-adjoint operator with discrete spectrum.
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- Other geometric operators:
  Length operator [Thiemann 1998; YM, Soo, Yang, 2010].
  $\hat{Q}$ operator [YM and Y. Ling, 2000].
  Quasi-local energy operator [Yang and YM, 2009].
Implementation of quantum constraints

- The Gaussian constraint operator can be defined in $\mathcal{H}_{kin}$. The kernel of the operator is the internal gauge invariant Hilbert space: $\mathcal{H}^G = \bigoplus_{\alpha,j} \mathcal{H}^G_{\alpha,j,l=0} \bigoplus \mathbb{C}$.

- One then naturally gets the gauge invariant spin-network basis $T_s$, $s = (\alpha(s), j_s, i_s)$ in $\mathcal{H}^G$:

  $$T_s = (\alpha,j,i) = \bigotimes_{v \in V(\alpha)} i_v \bigotimes_{e \in E(\alpha)} \pi^j_e(A(e)), \ (j_e \neq 0)$$

by assigning a non-trivial spin representation $j$ on each edge and a invariant tensor $i$ (intertwiner) on each vertex.
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In the strategy to solve the diffeomorphism constraint, the so-called group averaging technique is employed to obtain the diffeomorphism invariant Hilbert space [Ashtekar el, 1995].
The Quantum Dynamical Issues

- Hamiltonian constraint operators can be well defined in $H_{\text{kin}}$ or $H^G$ [Thiemann 1998]. Thus there is no UV divergence in the background independent quantum theory of gravity with diffeomorphism invariance.
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- The quantization technique for the Hamiltonian constraint can be generalized to quantize the Hamiltonian of matter fields coupled to gravity [Thiemann 1998].

- The result heightens our confidence that the issue of divergence in quantum fields theory can be cured in the framework of loop quantum gravity.
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The result heightens our confidence that the issue of divergence in quantum fields theory can be cured in the framework of loop quantum gravity.

To avoid possible quantum anomaly and find physical Hilbert space, master constraint programme was proposed [Thiemann 2003, 2005; Han and YM, 2005].

The Spinfoam path-integral formulation of LQG has been proposed [Reisenberger, Rovelli, 1996; Engle, Pereira, Rovelli, 2006; ...].
Action Principle of BDT

The most general action of BDT reads

\[
S[g, \phi] = \int_M d^4x \sqrt{-g} \left[ \frac{1}{2} (\phi R - \frac{\omega(\phi)}{\phi} (\partial_\mu \phi) \partial^\mu \phi) - \xi(\phi) \right]
\]  

(1)

where we set \(8\pi G = 1\), \(R\) denotes the scalar curvature of spacetime metric \(g_{\mu\nu}\), the coupling parameter \(\omega(\phi)\) and potential \(\xi(\phi)\) can be arbitrary functions of scalar field \(\phi\).
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• Note that \( f(R) \) theories can be regarded as special cases of above Brans-Dicke theories, since their action can be reformulated as

\[
S_f[g, \phi] = \frac{1}{2} \int_M d^4x \sqrt{-g} (\phi R - \xi(\phi)).
\]
A first-order action for BDT, which is equivalent to action (1) but can lead to a Hamiltonian connection formalism, reads [Zhou, Guo, Han, YM, 2013]

\[
S[e, \omega, \phi] = \int_M \frac{e}{2} \left( \phi e_i^a e_j^b \bar{\Omega}_{ab}^{IJ} - 2 e_i^a e_j^b \bar{\omega}_{a}^{IJ} \partial_b \phi + e_i^a e_j^b \bar{\partial}_a (e_b^c e^J_c \partial_c \phi) \\
+ \frac{\omega(\phi)}{\phi} (\bar{\partial}_a \phi) \bar{\partial}^a \phi - 2 \xi(\phi) + e_i^a e_j^b \frac{1}{\gamma} \bar{\Omega}_{ab}^{IJ} \right) d^4x,
\]

(2)

where \(e = det(e_i^a)\) is the determinant of the right-handed cotetrad \(e_i^a\), \(\bar{\Omega}_{ab}^{IJ} = \bar{\partial}_{[a} \bar{\omega}_{b]}^{IJ} + \bar{\omega}^{IK}_{[a} \bar{\omega}^{\phantom{IK}}_{b]K} J\) is the curvature of the \(SL(2, \mathbb{C})\) spin connection \(\bar{\omega}_{a}^{IJ}\), \(\ast\) denotes the Hodge dual of a differential form, and \(\gamma\) is an arbitrary real number.
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+ \frac{\omega(\phi)}{e} (\varpi^a_\phi) \varpi^a_\phi - 2\xi(\phi) + e^a_I e^b_J \gamma \tilde{\Omega}_{ab}^{IJ} \right) d^4x, \tag{2}
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- Another first-order action was proposed by [Cianfrani, Montani, 2009], which can give the Hamiltonian connection formalism of BDT in Einstein frame.
Connection Dynamics of BDT

- The detailed Hamiltonian analysis of action (2) leads to the connection dynamics of BDT in two sectors marked respectively by $\omega(\phi) \neq -\frac{3}{2}$ and $\omega(\phi) = -\frac{3}{2}$, coinciding with the results of canonical transformation from geometrical dynamics [Zhang, YM, 2011].
- The basic conjugate pairs consist of a $SU(2)$ connection $A^i_a$ and the densitized triad $E^b_j$, together with the scalar $\phi$ and its momentum $\pi$. 
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- The basic conjugate pairs consist of a $SU(2)$ connection $A^i_a$ and the densitized triad $E^b_j$, together with the scalar $\phi$ and its momentum $\pi$.

- In the sector of $\omega(\phi) \neq -3/2$, the Hamiltonian is a linear combination of first-class standard $SU(2)$ Gaussian constraint, diffeomorphism constraint and the following Hamiltonian constraint.

$$H = \frac{\phi}{2} \left[ F^j_{ab} - (\gamma^2 + \frac{1}{\phi^2})\varepsilon_{jmn}\tilde{K}^m_a\tilde{K}^n_b \right] \frac{\varepsilon_{jkl}E^a_k E^b_l}{\sqrt{h}}$$

$$+ \frac{1}{3 + 2\omega(\phi)} \left( \frac{(\tilde{K}^i_aE_i^a)^2}{\phi\sqrt{h}} + 2\frac{(\tilde{K}^i_aE_i^a)\pi}{\sqrt{h}} + \frac{\pi^2\phi}{\sqrt{h}} \right)$$

$$+ \frac{\omega(\phi)}{2\phi} \sqrt{h}(D_a\phi)D^a\phi + \sqrt{h}D_aD^a\phi + \sqrt{h}\xi(\phi), \quad (3)$$
The Special Sector of $\omega(\phi) = -3/2$

- In the special sector of $\omega(\phi) = -3/2$, an extra primary constraint appears as follows,

$$ S = \tilde{K}^i_a E^a_i - \phi \pi = 0, $$

which we call ”conformal” constraint since it generates spacetime conformal transformations.
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- \frac{3}{4\phi} \sqrt{h} (D_a \phi) D^a \phi + \sqrt{h} D_a D^a \phi + \sqrt{h} \xi(\phi). $$
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$$\quad - \frac{3}{4\phi} \sqrt{h}(D_a \phi) D^a \phi + \sqrt{h} D_a D^a \phi + \sqrt{h} \xi(\phi).$$

- The consistency of Hamiltonian evolution strongly restrict the potential of feasible BDT in this sector to only two cases: either $\xi(\phi) = 0$ or $\xi(\phi) = C \xi^2$. 
Polymer-like Representation of Scalar Field

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One extends the space $U$ of smooth scalar fields to the quantum configuration space $\bar{U}$ [Ashtekar, Lewandowski, Sahlmann, 2002].

A simple element $U \in \bar{U}$ may be thought as a point holonomy $U_\lambda = \exp(i\lambda \phi(x))$ at point $x \in \Sigma$, where $\lambda$ is a real number.

By Gel’fand-Naimark-Segal structure, there is also a unique diffeomorphism invariant measure $d\mu$ on $\bar{U}$. [Kaminski, Lewandowski, Bobienski, 2006]
Polymer-like Representation of Scalar Field

- Since the scalar field also reflects gravity, it is natural to employ the polymer-like representation for its quantization.
- One extends the space $\mathcal{U}$ of smooth scalar fields to the quantum configuration space $\bar{\mathcal{U}}$ [Ashtekar, Lewandowski, Sahlmann, 2002].
- A simple element $U \in \bar{\mathcal{U}}$ may be thought as a point holonomy $U_\lambda = \exp(i\lambda \phi(x))$ at point $x \in \Sigma$, where $\lambda$ is a real number.
- By Gel’fand-Naimark-Segal structure, there is also a unique diffeomorphism invariant measure $d\mu$ on $\bar{\mathcal{U}}$ [Kaminski, Lewandowski, Bobienski, 2006].
- Thus the kinematical Hilbert space of scalar field reads

$$\mathcal{H}_{\text{kin}}^{\text{sca}} = L^2(\bar{\mathcal{U}}, d\mu),$$

with the scalar-network basis $T_X(\phi) \equiv \prod_{x_j \in X} U_{\lambda_j}(\phi(x_j))$, where $X = \{x_1, \ldots, x_n\}$ is an arbitrary given set of finite number of points in $\Sigma$. 

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Polymer-like Representation of BDG

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- For the geometry sector, we have the unique diffeomorphism and internal gauge invariant representation for the quantum holonomy-flux algebra.
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- The quantum kinematics of LQG can be straightforwardly extended to BDT.
- For the geometry sector, we have the unique diffeomorphism and internal gauge invariant representation for the quantum holonomy-flux algebra.
- The total kinematical Hilbert space for the scalar-tensor gravity is a direct product $\mathcal{H}_{\text{kin}} := \mathcal{H}_{\text{geo}}^{\text{kin}} \otimes \mathcal{H}_{\text{sca}}^{\text{kin}}$ of geometric part and scalar field part.
Polymer-like Representation of BDG

- The quantum kinematics of LQG can be straightforwardly extended to BDT.
- For the geometry sector, we have the unique diffeomorphism and internal gauge invariant representation for the quantum holonomy-flux algebra.
- The total kinematical Hilbert space for the scalar-tensor gravity is a direct product $\mathcal{H}_{kin} := \mathcal{H}_{kin}^{geo} \otimes \mathcal{H}_{kin}^{sca}$ of geometric part and scalar field part.
- Both Gaussian constraint and diffeomorphism constraint can be solved at quantum level in the same way of LQG.
- The nontrivial task for loop quantum Brans-Dicke theories is to implement the Hamiltonian constraints at quantum level.
Quantum Dynamics

- In the sector of $\omega(\phi) = -3/2$, we promote the conformal constraint $S(\lambda)$ as a well-defined operator by acting on a given basis vector $T_{\alpha, x} \in \mathcal{H}_{\text{kin}}$ as

$$\hat{S}(\lambda) \cdot T_{\alpha, x} = \left( \sum_{v \in V(\alpha)} \frac{\lambda(v)}{\gamma^{3/2}(i\hbar)} [\hat{H}_E(1), \hat{V}_v] - \sum_{x \in X} \lambda(x) \hat{\phi}(x) \hat{\pi}(x) \right) \cdot T_{\alpha, x}.$$
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- By the regularization techniques developed for the Hamiltonian constraint operators of LQG [Thiemann, 1996], Polymer scalar field [Han, YM, 2006] and loop quantum $f(R)$ gravity [Zhang, YM, 2011], the Hamiltonian constraints in both sectors can be quantized as operators acting on cylindrical functions in $\mathcal{H}_{\text{kin}}$ in state-dependent ways.

To avoid possible quantum anomaly and find physical Hilbert space, master constraint method [Thiemann 2003; Han, YM, 2006] can also be applied to both sectors of quantum BDT [Zhang, YM, 2011].
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- Alternative Hamiltonian operators for LQC have been proposed, which have correct classical limit [Yang, Ding, YM, 2009].
- The big bang singularity of classical GR can be resolved by a quantum bounce of LQC [Ashtekar, Pawlowski, Singh, 2006].
Advances of LQC

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  - Alternative Hamiltonian operators for LQC have been proposed, which have correct classical limit [Yang, Ding, YM, 2009].
  - The big bang singularity of classical GR can be resolved by a quantum bounce of LQC [Ashtekar, Pawlowski, Singh, 2006].
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  - The inflationary scenario can be extend to Planck scale in LQC with a small window of novel effects. A detailed analysis of the LQC background geometry shows that the modes with $\lambda_{phy} \gtrsim R_{curv}$ excited in the pre-inflationary dynamics can be seen in the CMB within a narrow window in the $\phi_B$ parameter space of the theory [Agullo, Ashtekar, Nelson, 2013].
Symmetric Reduction

- Brans-Dicke cosmology: Spatially flat FRW universe
- The connections and the density weighted triads is reduced to [Ashtekar, Bojowald, Lewandowski, 2003]:
  \[ A^i_a = \tilde{c} V_o^{-(1/3)} \omega^i_a \quad \text{and} \quad E^a_i = p V_o^{-(2/3)} \sqrt{oq} \phi e^a_i. \]
- In the cosmological model of scalar-tensor theory, the fundamental Poisson brackets are given by:
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- By employing (4) one can show that \( \phi \) is monotonous wrt the cosmic time and hence can be viewed as an internal time.
Quantization Scheme

To quantize the model, in the geometrical sector, we employ the polymer-like representation of connection $\tilde{c}$ in $\mathcal{H}_{\text{kin}}^{\text{grav}} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$.

For a neat formulation of quantum dynamics, it is convenient to introduce new conjugate variables in the geometrical sector by a canonical transformation:

$$b := \sqrt{\Delta} \sqrt{|p|}, \quad \nu := \frac{4}{3} \sqrt{\Delta} \text{sgn}(p) |p|^{3/2},$$

where $\Delta \sim 4\sqrt{3} \pi \gamma l^2$ is the smallest non-zero eigenvalue of area operator in full LQG.
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In the kinematical Hilbert space $\mathcal{H}_{\text{kin}}^{\text{grav}}$, eigenstates of $\hat{\nu}$, which are labeled by real numbers $\nu$, constitute an orthonormal basis as:

$$\langle \nu_1 | \nu_2 \rangle = \delta_{\nu_1, \nu_2}.$$
Polymer-like representations

- For the quantization of the scalar field, we have two schemes: (i) Polymer-like representation; (ii) Schrodinger representation.

Polymer-like representation:

For the convenience of constructing a Hamiltonian constraint operator, we employ the polymer-like representation of the momentum $\pi$ of $\phi$.

One parameter ambiguity: Some small number $\lambda_0$ has to be introduced in order to define the operator $\hat{\pi} \equiv \frac{\sin(\lambda_0 \pi)}{\lambda_0}$.

In the kinematical Hilbert space $H_{\text{scalar}}$, eigenstates of $\hat{\phi}$, which are labeled by real numbers $\lambda$, constitute an orthonormal basis.

The operator corresponding to $\phi^{-1}$ can be defined as

$$\hat{\phi}^{-1} |\lambda\rangle = 4 \text{sgn}(\phi) (\lambda_0)^2 \hbar \left( |\lambda + \lambda_0|_1^2 - |\lambda|_1^2 \right) |\lambda\rangle =: D(\lambda) |\lambda\rangle.$$

Yongge Ma (BNU)
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The Dynamical Setting

- The Hamiltonian constraint for the $\omega \neq -3/2$ sector in the full theory of Brans-Dicke gravity reduces to 5 terms $H = \sum_{i=1}^{5} H_i$ for the spatially flat cosmology. They all can be quantized as following well-defined operators.
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- The actions of the corresponding terms of the Hamiltonian operator on a basis vector $|\lambda, \nu\rangle \equiv |\lambda\rangle \otimes |\nu\rangle$ read respectively as

$$
(\hat{H}_1 + \hat{H}_2)|\lambda, \nu\rangle = \frac{D(\lambda)}{2} \left( f_+(\nu)|\lambda, \nu + 4\rangle + f_0(\nu)|\lambda, \nu\rangle + f_-(\nu)|\lambda, \nu - 4\rangle \right),
$$

$$
\hat{H}_3|\lambda, \nu\rangle = -\frac{4\gamma\beta D(\lambda)}{\hbar^2} \left( \tilde{f}_+(\nu)|\lambda, \nu + 8\rangle - \tilde{f}_0(\nu)|\lambda, \nu\rangle + \tilde{f}_-(\nu)|\lambda, \nu - 8\rangle \right),
$$
Quantum Dynamics

\[ \hat{H}_4 |\lambda, v\rangle = -\frac{2\gamma^{1/2}\beta}{\lambda_0 \hbar} \left[ (B(v + 4) + B(v)) \times \right. \\
( f_+(v)|\lambda - \lambda_0, v + 4\rangle + f_+(v)|\lambda + \lambda_0, v + 4\rangle) \\
- \left. (B(v) + B(v - 4))(f_-(v)|\lambda - \lambda_0, v - 4\rangle - f_-(v)|\lambda + \lambda_0, v - 4\rangle) \right], \]

\[ \hat{H}_5 |\lambda, v\rangle = -\frac{\beta \hbar B(v)}{4\lambda_0^2} \left( (\lambda - \lambda_0)|\lambda - 2\lambda_0, v\rangle - 2\lambda|\lambda, v\rangle + (\lambda + \lambda_0)|\lambda + 2\lambda_0, v\rangle \right), \]

where \( \beta \equiv 3 + 2\omega. \)
Quantum Dynamics

\[ \hat{H}_4 |\lambda, \nu \rangle = -\frac{2\gamma^{1/2}\beta}{\lambda_0 \hbar} [ (B(\nu + 4) + B(\nu)) \times (f_+(\nu)|\lambda - \lambda_0, \nu + 4 \rangle + f_+(\nu)|\lambda + \lambda_0, \nu + 4 \rangle) 
- (B(\nu) + B(\nu - 4))(f_-(\nu)|\lambda - \lambda_0, \nu - 4 \rangle - f_-(\nu)|\lambda + \lambda_0, \nu - 4 \rangle)], \]

\[ \hat{H}_5 |\lambda, \nu \rangle = -\frac{\beta \hbar B(\nu)}{4\lambda_0^2} \left( (\lambda - \lambda_0) |\lambda - 2\lambda_0, \nu \rangle - 2\lambda |\lambda, \nu \rangle + (\lambda + \lambda_0) |\lambda + 2\lambda_0, \nu \rangle \right), \]

where \( \beta \equiv 3 + 2\omega. \)

- We can show by semiclassical analysis that the above Hamiltonian operator has correct classical limit.
Polymer-like plus Schrodinger representations

If we employ the Schrodinger representation of the scalar field, the Hamiltonian operator for Brans-Dicke cosmology can also be well defined on the coupled Hilbert space.

\[
H_F = -\sqrt{3} \Delta^2 \gamma^2 \kappa \phi |v| \sin^2 b + 2 \sqrt{3} \kappa \beta \Delta^3 / 2 |v| \phi (3 \hbar^4 \sin(b)v + \pi \phi)^2 + \Delta^3 / 2 |v|^2 \sqrt{3} \rho,
\]

where \( \rho \) is the energy density of minimally coupled matter field.

This effective Hamiltonian constraint of loop quantum Brans-Dicke cosmology gives the modified evolution equation:

\[
(\dot{a} a + \dot{\phi} \phi)^2 = (1 - \rho e^\rho c) + \dot{\phi} \phi (1 - \sqrt{1 - \rho e^\rho c})^2,
\]

(5)

where \( \rho e \equiv \beta \dot{\phi}^2 / 4 \kappa + \phi \rho \) and \( \rho c \equiv 3 \gamma^2 \Delta \kappa \).
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- If we employ the Schrodinger representation of the scalar field, the Hamiltonian operator for Brans-Dicke cosmology can also be well defined on the coupled Hilbert space.

- In this scheme, we can obtain the effective Hamiltonian constraint as

\[
H_F = -\frac{\sqrt{3}\Delta}{2\gamma^2\kappa\phi}|v|\sin^2 b + \frac{2\sqrt{3}\kappa}{\beta\Delta^{3/2}|v|\phi} \left(\frac{3\hbar}{4} \sin(b)v + \pi\phi\right)^2 + \frac{\Delta^{3/2}|v|}{2\sqrt{3}}\rho,
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where \(\rho\) is the energy density of minimally coupled matter field.
Polymer-like plus Schrödinger representations

- If we employ the Schrödinger representation of the scalar field, the Hamiltonian operator for Brans-Dicke cosmology can also be well defined on the coupled Hilbert space.
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\[
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\left(\frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi}\right)^2 = \left(\frac{1}{\phi} \sqrt{\frac{\kappa}{3}} \rho_e (1 - \frac{\rho_e}{\rho_c}) + \frac{\phi}{2\phi} (1 - \sqrt{1 - \frac{\rho_e}{\rho_c}})\right)^2,
\]

where \(\rho_e \equiv \frac{\beta\dot{\phi}^2}{4\kappa} + \phi\rho\) and \(\rho_c \equiv \frac{3}{\gamma^2\Delta\kappa}\).
Quantum Bounce of Loop Quantum Brans-Dicke Cosmology

[Zhang, Artymowski, YM, 2013]

FIG. 1. Left and right panels present the evolution of the Hubble parameter in the Planck units as a function of the Brans-Dicke field (left panel, vacuum solution) or the massless scalar field (right panel, massless scalar field domination) for realistic values of $\beta$. Initial conditions are chosen to be: $\phi_{cr} = 1$ (left panel) and $\frac{\dot{\phi}_{cr}}{\phi_{cr}} = -\frac{\phi_{cr}}{\beta}$, $\varphi_{cr} - \frac{\varphi_{cr}^2}{\beta} - \frac{\phi_{cr}}{\phi_{cr}^2} \varphi_{cr} = M_{Pl}$ (right panel).
Conclusions

- The Hamiltonian connection formulation of Brans-Dicke gravity has been derived from their Lagrangian formulation. Two sectors of BDT are marked off by the coupling parameter $\omega(\phi)$.
- In the sector of $\omega(\phi) = -\frac{3}{2}$, the feasible theories are restricted and a new primary constraint generating conformal transformations of spacetime is obtained.
Conclusions

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- Due to the $su(2)$-connection dynamical formalism, the 4-dimensional BDT have been quantized by extending LQG scheme.
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- Due to the $su(2)$-connection dynamical formalism, the 4-dimensional BDT have been quantized by extending LQG scheme.
- The non-perturbative loop quantization procedure is valid for a rather general class of metric theories of gravity.
- The symmetry-reduced model of loop quantum Brans-Dicke cosmology is set up for the case $\omega \neq -3/2$.
- The scalar may play the role of emergent (internal) time. The notion of time emerges from gravity itself rather than an outside matter field.
Conclusions

- Scheme (i): polymer-like representations
  - The polymer representation of the momentum $\pi$ of the scalar field is more convenient for the purpose of constructing manageable Hamiltonian constraint operator.
  - An one-parameter ambiguity appears in this construction for the scalar sector, which is also the feature of the full theory.
  - The Hamiltonian constraint operator gives rise to a difference equation representing a discrete evolution of the universe.
  - In contrast to the old treatment of LQC, both space and (internal) time are "discrete" in this treatment of loop quantum BD cosmology.
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  - In contrast to the old treatment of LQC, both space and (internal) time are "discrete" in this treatment of loop quantum BD cosmology.

- Scheme (ii): polymer-like plus Schrodinger representations
  - The Hamiltonian constraint operator, effective Hamiltonian constraint and modified cosmological evolution equation are obtained.
  - The classical big bang singularity of Brans-Dicke cosmology is again replaced by a quantum bounce.
The loop quantum cosmological model can be used to show the inequivalence between Jordan and Einstein frames for the quantization of Brans-Dicke theories [Artymowski, YM, Zhang, 2013].

The method proposed by [Bodendorfer, Thiemann, Thurn, 2011] for the loop quantization of higher dimensional GR can also be extended to higher (> 4) dimensional Brans-Dicke theories. [Han, YM, Zhang, 2013]
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Applications to black holes are desirable [Guo, Yang, YM...; related work see e.g. Bodendorfer 2013].

It is also desirable to quantize metric theories of gravity by the covariant spin foam approach [Zhou, YM...].
Summary and Outlook

Outlook

- The loop quantum cosmological model can be used to show the inequivalence between Jordan and Einstein frames for the quantization of Brans-Dicke theories [Artymowski, YM, Zhang, 2013].
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- Applications to black holes are desirable [Guo, Yang, YM...; related work see e.g. Bodendorfer 2013].
- It is also desirable to quantize metric theories of gravity by the covariant spin foam approach [Zhou, YM...].
- In full theory of loop quantum Brans-Dicke gravity, can we somehow use the scalar field to deparameterize the system and isolate physical Hamiltonian? [Giesel, Han, YM...]
References

- Y. Han, Y. Ma and X. Zhang, Connection dynamics for higher dimensional scalar-tensor theories of gravity, arXiv:1304.0209.