Quantum entropy of Kerr-de Sitter black hole due to arbitrary spin fields

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Content:

I. Introduction: (Backgrounds and Motivations, Summary of Previous Results)

Bekenstein-Hawking formula:

$$S = \frac{A}{4}\,, \qquad T = \frac{\kappa}{2\pi} \qquad (1)$$

Statistical, quantum, or dynamic origin of black hole entropy ?!

- Statistical origin of the black hole entropy — Brick Wall Method (BWM) by 't Hooft.

$$\Phi(x) = 0 \quad \text{for} \quad \begin{cases} r \geq L\,, \\ r \leq r_h + \varepsilon\,. \end{cases} \qquad (2)$$

In this model, the black hole entropy is identified with the statistical-mechanical entropy of a thermal gas of quantum field excitations outside the event horizon.

- Wide applications of BWM:

1) **Various static spherically symmetric black holes**: (Schwarzschild, Reissner-Nordström (RN), dilaton, axion, dyon, Schwarzschild-de Sitter, ...) for **all known species of particles**: scalar, spinor, electromagnetic and arbitrary spin fields, etc.

$$S = g_s \frac{15 + (-1)^{2s}}{16} \Big[\frac{A_h}{48\pi\epsilon^2} + \frac{C}{45} \ln\Big(\frac{\Lambda}{\epsilon}\Big) \Big] \quad (3)$$

$C = 1$ for Schwarzschild

$C = 1 - \frac{3Q^2}{2r_h^2}$ for RN

$C = 1 + \frac{3(Q^2 + P^2)}{2(r_h^2 - D^2)}$ for GM dilaton

$C = 1 - \frac{3Q^2}{4r_h^2} e^{-2\phi_0}$ for GHS dilaton

$C = 1 - \frac{3Q^2}{2r_h^2} e^{-2\alpha\phi_0}$ for GH dilaton

However all these calculations did not consider the contribution of the subleading term from the coupling of the spin of particles with the rotation of black holes.

2) **Rotating axisymmetric black holes**: Entropy of scalar fields for Kerr, Kerr-Newman, Kerr-Sen (EMDA), Klein-Klauza black hole.

Entropy of **Dirac spinor** fields for Kerr-Newman black hole.

Liu & Zhao, **PRD 61** (2000) 063003.

Recently much attention has been paid to the contribution to the quantum entropies of black holes due to higher spin fields.

Entropy of Kerr black hole

$$\frac{S}{g_s} = \frac{15+(-1)^{2s}}{16}\Big\{\frac{A_h}{48\pi\epsilon^2} + \frac{1}{45}\Big[1 - \frac{3Q^2}{4r_h^2} \cdot\Big(1 + \frac{r_h^2 + a^2}{ar_h}\arctan\big(\frac{a}{r_h}\big)\Big)\Big]\ln\Big(\frac{\Lambda}{\epsilon}\Big)\Big\} + \frac{3+(-1)^{2s}}{4}\cdot\frac{s^2}{12}C'\ln\Big(\frac{\Lambda}{\epsilon}\Big) \qquad (4)$$

due to massless **Dirac**, **electromagnetic**,

Jing & Yan, **PRD 63** (2001) 084028

and linearized **gravitational** fields,

Jing & Yan, **PRD 64** (2001) 064015

$$C' = 1 - \frac{r_h^2 + a^2}{a r_h} \arctan\left(\frac{a}{r_h}\right)$$

as well as Rarita-Schwinger field,
Jing, **CPL 20** (2001) 459 ;
López-Ortega, **GRG 35** (2003) 59

$$C' = \frac{a^2 - 5r_h^2}{4r_h^2} + \frac{5r_h^2 + a^2}{4r_h^2} \cdot \frac{r_h^2 + a^2}{a r_h} \arctan\left(\frac{a}{r_h}\right)$$

Jing & Yan showed that the contribution of the spins to the logarithmic terms shall decrease the entropy of a Kerr black hole. López-Ortega pointed out that the entropy is increased by the logarithmic terms relating to the square of spins of particles.

How the spins of the quantum field changes the quantum entropy of a rotating black hole is an interesting question and deserves to be further clarified.

- Drawbacks of original BWM:

a) little mass approximation;
b) neglecting logarithm term;
c) taking the L^3 term as a contribution of the vacuum surrounding the black hole;
d) impossible to apply to the case of black holes with two horizons.

Improved thin-layer BWM:

By taking only the entropy of a thin layer near the event horizon of a black hole into account.

The Kerr-de Sitter black hole has a cosmological horizon and a black hole event horizon, it is a thermal nonequilibrium system that the temperature of the two horizons is different from one another. In principle, each horizon can be treated as an isolated thermodynamical system.

Although the total system consisted of the two horizons is thermal nonequilibrium, the thin layer near the horizon can be taken as a local thermal equilibrium system. The quantum entropy of such a black hole can be calculated via this improved BWM which means that the entropy comes from a thin layer near the horizon, the entropy is then taken the sum of the contribution from each horizon of the considered spacetime.

Two main motivations for studying black holes with a cosmological constant:

(a) dS/CFT correspondence;

(b) Recent astrophysical observations of type Ia supernovae which indicates a positive cosmological constant. A realistic black hole may be in an asymptotically de Sitter (nonflat) space.

It becomes important to investigate the **effect** of the cosmological constant on the entropies of these kinds of black holes.

In this study, the entropies of Kerr-de Sitter black holes due to higher spin fields are taken up for consideration on which the effects of the cosmological constant and that of the spins of particles are emphasized.

The purpose is to deduce expressions of the entropy of Kerr-de Sitter black holes arising from arbitrary spin fields by using the improved thin-layer BWM and to investigate effects of the spins of particles and the cosmological constant on the statistical entropy.

II. Perturbations of spin fields in the Kerr-de Sitter space

The line element Kerr-de Sitter spacetime can be written in a Boyer-Lindquist type of coordinate system as

$$ds^2 = -\frac{\Delta_r}{\chi^2\Sigma}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\Delta_\theta \sin^2\theta}{\chi^2\Sigma}\left[adt - (r^2 + a^2)d\varphi\right]^2 + \Sigma\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right), \qquad (5)$$

where

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{r^2}{l^2}\right) - 2Mr\,,$$

$$\Delta_\theta = 1 + \frac{a^2}{l^2}\cos^2\theta\,, \qquad \chi = 1 + \frac{a^2}{l^2}\,,$$

$$\Sigma = \rho\rho^* = r^2 + a^2\cos^2\theta\,,$$

$$\rho = r + ia\cos\theta\,, \qquad \rho^* = r - ia\cos\theta\,.$$

Cosmological constant, metric determinant, Ricci scalar, nonvanishing Weyl scalar:

$$\Lambda = 3/l^2; \qquad \sqrt{-g} = \Sigma\sin\theta/\chi^2;$$

$$\mathcal{R} = 4\Lambda = 12/l^2; \qquad \Psi_2 = -M/\rho^{*3}.$$

Null-tetrad vectors (Newman-Penrose formalism)

$$l^\mu = \frac{1}{\Delta_r}\Big[(r^2+a^2)\chi, \Delta_r, 0, a\chi\Big],$$
$$n^\mu = \frac{1}{2\Sigma}\Big[(r^2+a^2)\chi, -\Delta_r, 0, a\chi\Big],$$
$$m^\mu = \frac{1}{\sqrt{2\Delta_\theta}\rho}\Big(i\chi a\sin\theta, 0, \Delta_\theta, \frac{i\chi}{\sin\theta}\Big),$$
$$\overline{m}^\mu = \frac{1}{\sqrt{2\Delta_\theta}\rho^*}\Big(-i\chi a\sin\theta, 0, \Delta_\theta, \frac{-i\chi}{\sin\theta}\Big),$$

nonvanishing spin coefficients:

$$\tilde{\rho} = \frac{-1}{\rho^*}, \quad \mu = \frac{-\Delta_r}{2\Sigma\rho^*}, \quad \gamma = \mu + \frac{\Delta_r'}{4\Sigma},$$
$$\tau = \frac{-ia\sqrt{\Delta_\theta}\sin\theta}{\sqrt{2}\Sigma}, \quad \pi = \frac{ia\sqrt{\Delta_\theta}\sin\theta}{\sqrt{2}\rho^{*2}}, \quad (6)$$
$$\beta = \frac{\sqrt{\Delta_\theta}}{2\sqrt{2}\rho}\Big(\cot\theta + \frac{\Delta_\theta'}{2\Delta_\theta}\Big), \quad \alpha = \pi - \beta^*,$$

where a prime denotes the partial differential with respect to its argument.

Directional derivatives: $(\sim e^{i(m\varphi-\omega t)})$

$$D = \mathcal{D}_0 \,, \qquad \Delta = \frac{-\Delta_r}{2\Sigma}\mathcal{D}_0^\dagger \,,$$
$$\delta = \frac{\sqrt{\Delta_\theta}}{\sqrt{2}\rho}\mathcal{L}_0^\dagger \,, \quad \bar{\delta} = \frac{\sqrt{\Delta_\theta}}{\sqrt{2}\rho^*}\mathcal{L}_0 \,, \qquad (7)$$

where

$$\mathcal{D}_n = \frac{\partial}{\partial r} - \frac{i\chi K_1}{\Delta_r} + n\frac{\Delta_r'}{\Delta_r} \,,$$
$$\mathcal{D}_n^\dagger = \frac{\partial}{\partial r} + \frac{i\chi K_1}{\Delta_r} + n\frac{\Delta_r'}{\Delta_r} \,,$$
$$\mathcal{L}_n = \frac{\partial}{\partial \theta} - \frac{\chi K_2}{\Delta_\theta} + n\Big(\cot\theta + \frac{\Delta_\theta'}{2\Delta_\theta}\Big) \,,$$
$$\mathcal{L}_n^\dagger = \frac{\partial}{\partial \theta} + \frac{\chi K_2}{\Delta_\theta} + n\Big(\cot\theta + \frac{\Delta_\theta'}{2\Delta_\theta}\Big) \,,$$
$$K_1 = \omega(r^2 + a^2) - ma \,,$$
$$K_2 = a\omega \sin\theta - \frac{m}{\sin\theta} \,.$$

It can be shown that perturbation master equations in the Kerr-de Sitter geometry are separable for massless **Klein-Gordon scalar** ($s = 0$), Weyl neutrino ($s = 1/2$), Maxwell electromagnetic ($s = 1$), **Rarita-Schwinger gravitino** ($s = 3/2$), and linearized Einstein gravitational ($s = 2$) fields. The **Teukolsky's master equations** controlling the perturbations of Kerr-de Sitter black hole for massless arbitrary spin $s = 1/2, 1, 3/2,$ and 2 fields reads

$$\Big\{[D - (2s-1)\epsilon + \epsilon^* - 2s\tilde{\rho} - \tilde{\rho}^*](\Delta - 2s\gamma + \mu) - [\delta - (2s-1)\beta - \alpha^* - 2s\tau + \pi^*](\bar{\delta} - 2s\alpha + \pi) - (s-1)(2s-1)\Psi_2\Big\}\Phi_s = 0\,, \tag{8}$$

for spin weight $s = 1/2, 1, 3/2, 2$ and

$$\Big\{[\Delta + (2s-1)\gamma - \gamma^* + 2s\mu + \mu^*](D + 2s\epsilon - \tilde{\rho}) - [\bar{\delta} + (2s-1)\alpha + \beta^* + 2s\pi - \tau^*](\delta + 2s\beta - \tau) - (s-1)(2s-1)\Psi_2\Big\}\Phi_{-s} = 0\,, \tag{9}$$

for spin weight $s = -1/2, -1, -3/2, -2$.

Eqs. (8) and (9) are also valid when $s = 0$, they coincide with the massless (minimal conformal coupling) Klein-Gordon scalar field equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\Big(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\Phi\Big) + \frac{1}{6}\mathcal{R}\Phi = 0\,, \tag{10}$$

with $\Phi = \Phi_0 = \Phi_{-0}$.

All the above equations are separable and can be written as (ignoring $e^{i(m\varphi-\omega t)}$)

$$\Big[\frac{1}{\Sigma}\Big(\Delta_r\mathcal{D}_1\mathcal{D}_s^\dagger + \sqrt{\Delta_\theta}\mathcal{L}_{1-s}^\dagger\sqrt{\Delta_\theta}\mathcal{L}_s\Big) +2(2s-1)\Big(\frac{i\chi\omega}{\rho^*} - \frac{s-1}{l^2}\Big)\Big]\Phi_s = 0\,, \tag{11}$$

$$\Big[\frac{1}{\Sigma}\Big(\Delta_r\mathcal{D}_{1-s}^\dagger\mathcal{D}_0 + \sqrt{\Delta_\theta}\mathcal{L}_{1-s}\sqrt{\Delta_\theta}\mathcal{L}_s^\dagger\Big) -2(2s-1)\Big(\frac{i\chi\omega}{\rho^*} + \frac{s-1}{l^2}\Big)\Big](\rho^{*2s}\Phi_{-s}) = 0\,. \tag{12}$$

They are also satisfied by the scalar Debye potentials $\phi_s = \Phi_s/\rho^{*2s}$ and $\phi_{-s} = \rho^{*2s}\Phi_{-s}$.

From their obvious expressions

$$\frac{1}{\Sigma}\Big\{\Delta_r^{-s}\frac{\partial}{\partial r}\Big(\Delta_r^{1+s}\frac{\partial}{\partial r}\Big) + \frac{\chi^2K_1^2 - is\chi K_1\Delta_r'}{\Delta_r}$$
$$+\frac{s}{2}\Delta_r'' + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\Delta_\theta\sin\theta\frac{\partial}{\partial\theta}\Big)$$
$$-\frac{1}{\Delta_\theta}\Big[\chi K_2 - s\Big(\frac{\Delta_\theta'}{2} + \Delta_\theta\cot\theta\Big)\Big]^2$$
$$+4is\chi\omega\rho - \frac{4s^2+2}{l^2}\Sigma\Big\}\Phi_s = 0\,, \quad (13)$$

and

$$\frac{1}{\Sigma}\Big\{\Delta_r^{s}\frac{\partial}{\partial r}\Big(\Delta_r^{1-s}\frac{\partial}{\partial r}\Big) + \frac{\chi^2K_1^2 + is\chi K_1\Delta_r'}{\Delta_r}$$
$$-\frac{s}{2}\Delta_r'' + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\Delta_\theta\sin\theta\frac{\partial}{\partial\theta}\Big)$$
$$-\frac{1}{\Delta_\theta}\Big[\chi K_2 + s\Big(\frac{\Delta_\theta'}{2} + \Delta_\theta\cot\theta\Big)\Big]^2$$
$$-4is\chi\omega\rho - \frac{4s^2+2}{l^2}\Sigma\Big\}\phi_{-s} = 0\,, \quad (14)$$

one can easily find that they are dual by interchanging $s \to -s$.

Eqs. (13) and (14) can be combined into the form of Teukolsky's master equation

$$\Big\{\frac{\Delta_r}{\Sigma}\frac{\partial^2}{\partial r^2}+\frac{(1+s)\Delta_r'}{\Sigma}\frac{\partial}{\partial r}+\frac{s}{2\Sigma}\Delta_r''+\frac{\Delta_\theta}{\Sigma}\frac{\partial^2}{\partial\theta^2}$$
$$+\frac{\Delta_\theta'+\Delta_\theta\cot\theta}{\Sigma}\frac{\partial}{\partial\theta}+\frac{\omega^2\chi^2}{\Sigma}\Big[\frac{(r^2+a^2)^2}{\Delta_r}$$
$$-\frac{a^2\sin^2\theta}{\Delta_\theta}\Big]-\frac{2\omega m a\chi^2}{\Sigma}\Big(\frac{r^2+a^2}{\Delta_r}-\frac{1}{\Delta_\theta}\Big)$$
$$+\frac{m^2\chi^2}{\Sigma}\Big(\frac{a^2}{\Delta_r}-\frac{1}{\Delta_\theta\sin^2\theta}\Big)+\frac{2s\omega\chi}{\Sigma}\Big[2ir$$
$$-\frac{i\Delta_r'}{2\Delta_r}(r^2+a^2)+a\sin\theta\Big(\frac{\Delta_\theta'}{2\Delta_\theta}-\cot\theta\Big)\Big]$$
$$+\frac{2sm\chi}{\Sigma}\Big[\frac{ia\Delta_r'}{2\Delta_r}-\frac{1}{\sin\theta}\Big(\frac{\Delta_\theta'}{2\Delta_\theta}+\cot\theta\Big)\Big]$$
$$-\frac{4s^2+2}{l^2}-\frac{s^2\Delta_\theta}{\Sigma}\Big(\frac{\Delta_\theta'}{2\Delta_\theta}+\cot\theta\Big)^2\Big\}\Phi_s=0\,,$$
$$(s=0,\pm1/2,\pm1,\pm3/2,\pm2). \qquad (15)$$

III. Improved thin-Layer BWM

Distinguished from the original BWM, the thin-layer BWM suggests that the entropy of a black hole with two horizons mainly comes from a very thin layer in the vicinity of the horizon where exists a **local thermal equilibrium**. Just as the original BWM, it impose a small ultraviolet cutoff ε such that

$$\Psi(x) = 0 \qquad \text{for} \quad r \leq r_h + \varepsilon\,, \tag{16}$$

where r_h denotes one horizon of the Kerr-de Sitter black hole, satisfying

$$\Delta_{r_h} = (r_h^2 + a^2)(1 - \frac{r_h^2}{l^2}) - 2Mr_h = 0\,.$$

To remove the infrared divergence, it introduce another cutoff parameter — an arbitrary big integer N such that

$$\Psi(x) = 0 \qquad \text{for} \quad r \geq r_h + N\varepsilon\,. \tag{17}$$

Improved thin-layer BWM boundary condition:

$$L \Longrightarrow r_h + N\varepsilon$$

A crucial difference from the original BWM.

1) The Kerr-de Sitter black hole has a cosmological horizon and a black hole event horizon, there exist no thermal equilibrium over the entire spacetime since the two horizons have different temperatures. 2) There can not exist a global thermal equilibrium between the external field and the hole in a large spatial region, and statistical physics laws become invalid in this case.

However the global thermal equilibrium is not needed, the notion of local thermal equilibrium is still worked very well and is crucial to the discussion. In this thin-layer BWM, it is assumed that the total entropy is mainly attributed to the two thin-layers near the two horizons, namely it is a linear sum of the entropy of the black hole horizon and that of the cosmological horizon.

IV. Entropy of Kerr-de Sitter black holes due to arbitrary spin fields

WKB approximation: $\Phi_s \sim e^{i(k_r r + k_\theta \theta)}$

$$\frac{\Delta_r}{\Sigma}k_r^2 + \frac{\Delta_\theta}{\Sigma}k_\theta^2 + \frac{\omega^2\chi^2}{\Sigma}\Big[\frac{a^2\sin^2\theta}{\Delta_\theta} - \frac{(r^2+a^2)^2}{\Delta_r}\Big] + \frac{2\omega m a\chi^2}{\Sigma}\Big(\frac{r^2+a^2}{\Delta_r} - \frac{1}{\Delta_\theta}\Big) - \frac{m^2\chi^2}{\Sigma}\Big(\frac{a^2}{\Delta_r} - \frac{1}{\Delta_\theta\sin^2\theta}\Big) + \frac{2s\omega\chi a\sin\theta}{\Sigma}\Big(\cot\theta - \frac{\Delta'_\theta}{2\Delta_\theta}\Big) + \frac{2sm\chi}{\Sigma\sin\theta}\Big(\frac{\Delta'_\theta}{2\Delta_\theta} + \cot\theta\Big) + \frac{4s^2+2}{l^2} + \frac{s^2\Delta_\theta}{\Sigma}\Big(\frac{\Delta'_\theta}{2\Delta_\theta} + \cot\theta\Big)^2 - \frac{s}{2\Sigma}\Delta''_r = 0\,, \quad (18)$$

Rewrite it as

$$\frac{k_r^2}{g_{rr}} + \frac{k_\theta^2}{g_{\theta\theta}} + \frac{g_{\varphi\varphi}\omega^2 + 2g_{t\varphi}m\omega + g_{tt}m^2}{\mathcal{D}} + 2(\omega B + mC) + H_s = 0\,, \quad (19)$$

where

$$g_{rr} = \frac{\Sigma}{\Delta_r}\,, \qquad g_{tt} = \frac{\Delta_\theta a^2 \sin^2\theta - \Delta_r}{\chi^2 \Sigma}\,,$$

$$g_{\theta\theta} = \frac{\Sigma}{\Delta_\theta}\,, \qquad g_{t\varphi} = \frac{\Delta_r - (r^2 + a^2)\Delta_\theta}{\chi^2 \Sigma} a \sin^2\theta\,,$$

$$g_{\varphi\varphi} = \frac{(r^2 + a^2)^2 \Delta_\theta - \Delta_r a^2 \sin^2\theta}{\chi^2 \Sigma} \sin^2\theta\,,$$

$$\mathcal{D} = g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2 = -\frac{\Delta_r \Delta_\theta \sin^2\theta}{\chi^4}\,,$$

$$B = \frac{s\chi a \sin\theta}{\Sigma}\Big(\cot\theta - \frac{\Delta'_\theta}{2\Delta_\theta}\Big)\,,$$

$$C = \frac{s\chi}{\Sigma \sin\theta}\Big(\frac{\Delta'_\theta}{2\Delta_\theta} + \cot\theta\Big)\,,$$

$$H_s = \frac{s^2 \Delta_\theta}{\Sigma}\Big(\frac{\Delta'_\theta}{2\Delta_\theta} + \cot\theta\Big)^2 + \frac{4s^2 + 2}{l^2} - \frac{s}{2\Sigma}\Delta''_r\,.$$

Suppose that a quantum field in a thin-layer very near the event horizon is in local thermal equilibrium with the Kerr-de Sitter black hole at temperature $1/\beta$, it is appropriate to assume that the quantum field is rotating with an angular velocity Ω_h in this thin-layer. After making substitution $\mathcal{E} = \omega - m\Omega_h$, Eq. (19) is reduced to

$$\frac{k_r^2}{g_{rr}} + \frac{g_{\varphi\varphi}\mathcal{E}^2 + 2(g_{t\varphi} + g_{\varphi\varphi}\Omega_h)m\mathcal{E} + \tilde{g}_{tt}m^2}{\mathcal{D}} + \frac{k_\theta^2}{g_{\theta\theta}} + 2[\mathcal{E}B + m(\Omega_h B + C)] + H_s = 0\,, \tag{20}$$

and can be rewritten as

$$\frac{k_r^2}{g_{rr}} + \frac{k_\theta^2}{g_{\theta\theta}} + \frac{-\tilde{g}_{tt}}{-\mathcal{D}}\big(m + m_0\big)^2 = \frac{1}{-\tilde{g}_{tt}}\big(\mathcal{E} + sW\big)^2 - V_s\,, \tag{21}$$

$\tilde{g}_{tt}$: the temporal component of the metric of the dragged optical space, W: the "spin potential", V_s: the effective potential:

$$
\begin{aligned}
\tilde{g}_{tt} &= g_{tt} + 2g_{t\varphi}\Omega_h + g_{\varphi\varphi}\Omega_h^2\,, \\
\tilde{g}_{tt} m_0 &= (g_{t\varphi} + g_{\varphi\varphi}\Omega_h)\mathcal{E} + (\Omega_h B + C)\mathcal{D}\,, \\
sW &= (g_{tt} + g_{t\varphi}\Omega_h)B - (g_{t\varphi} + g_{\varphi\varphi}\Omega_h)C\,, \\
V_s &= H_s - (g_{tt}B^2 - 2g_{t\varphi}BC + g_{\varphi\varphi}C^2) \\
&= P - \frac{s\Delta_r''}{2\Sigma}\,. \qquad (22)
\end{aligned}
$$

For a given energy $\omega = \mathcal{E} + m\Omega_h$ and a fixed m, the total number of modes with energy less than ω is equal to the number of states in the classical phase space

$$
\Gamma(\mathcal{E}, s) = \frac{1}{3\pi}\int d\theta \int_{r_h+\varepsilon}^{r_h+N\varepsilon} dr \frac{\sqrt{-g}}{(-\tilde{g}_{tt})^2} \times \Big[(\mathcal{E} + sW)^2 + \tilde{g}_{tt}V_s\Big]^{3/2}\,, \qquad (23)
$$

provided that

$$
g_{rr} > 0\,,\ \ g_{\theta\theta} > 0\,,\ -\tilde{g}_{tt} > 0\,,\ \ -\mathcal{D} > 0\,, \quad (\mathcal{E} + sW)^2 + \tilde{g}_{tt}V_s \geq 0\,. \qquad (24)
$$

Summing over the positive and negative helicity states $p = \pm s$, we get the total states number (density of states)

$$\begin{aligned}
\Gamma(\mathcal{E}) &= \frac{g_s}{2}\Big[\Gamma(\mathcal{E}, s) + \Gamma(\mathcal{E}, -s)\Big] \\
&= \frac{g_s}{6\pi}\int d\theta \int_{r_h+\varepsilon}^{r_h+N\varepsilon} dr \frac{\sqrt{-g}}{(-\tilde{g}_{tt})^2} \\
&\quad \times \Big\{\Big[(\mathcal{E} + sW)^2 + \tilde{g}_{tt}V_s\Big]^{3/2} \\
&\quad + \Big[(\mathcal{E} - sW)^2 + \tilde{g}_{tt}V_{-s}\Big]^{3/2}\Big\} \\
&\approx \frac{g_s}{3\pi}\int d\theta \int_{r_h+\varepsilon}^{r_h+N\varepsilon} dr \frac{\sqrt{-g}}{(-\tilde{g}_{tt})^2}\Big[\mathcal{E}^3 + \frac{3}{2}\tilde{g}_{tt}P\mathcal{E}\Big] \\
&\equiv \frac{g_s}{3\pi}\Big(I_1\mathcal{E}^3 + 3I_2\mathcal{E}\Big)\,, \qquad (25)
\end{aligned}$$

where two integrals are defined by,

$$\begin{aligned}
I_1 &= \int d\theta \int_{r_h+\varepsilon}^{r_h+N\varepsilon} dr \frac{\sqrt{-g}}{\tilde{g}_{tt}^2}\,, \\
I_2 &= \frac{1}{2}\int d\theta \int_{r_h+\varepsilon}^{r_h+N\varepsilon} dr \frac{\sqrt{-g}}{\tilde{g}_{tt}} P\,, \\
P &= \frac{4s^2+2}{l^2} + \frac{s^2\Delta_\theta}{\Sigma}\Big(\frac{\Delta_\theta'}{2\Delta_\theta} + \cot\theta\Big)^2 \\
&\quad -(g_{tt}B^2 - 2g_{t\varphi}BC + g_{\varphi\varphi}C^2)\,. \qquad (26)
\end{aligned}$$

For a local (quasi-)equilibrium ensemble of states of spin fields, the **free energy** can be expressed as

$$
\begin{aligned}
F &= -\int_0^\infty d\mathcal{E} \frac{\Gamma(\mathcal{E})}{e^{\beta\mathcal{E}} - (-1)^{2s}}, \\
&\approx -\frac{g_s}{3\pi} \int_0^\infty \frac{d\mathcal{E}}{e^{\beta\mathcal{E}} - (-1)^{2s}} \left[I_1 \mathcal{E}^3 + 3 I_2 \mathcal{E} \right] \\
&= -g_s \Big[2\zeta(4) \frac{15 + (-1)^{2s}}{16\pi\beta^4} I_1 \\
&\quad + \zeta(2) \frac{3 + (-1)^{2s}}{4\pi\beta^2} I_2 \Big]. \qquad (27)
\end{aligned}
$$

The entropy of the Kerr-de Sitter black hole due to arbitrary spin fields is obtained from the standard formula $S = \beta^2(\partial F/\partial\beta)$,

$$
\begin{aligned}
S = \frac{g_s}{2\pi} \Big[&\zeta(4) \frac{15 + (-1)^{2s}}{\beta^3} I_1 \\
&+ \zeta(2) \frac{3 + (-1)^{2s}}{\beta} I_2 \Big]. \qquad (28)
\end{aligned}
$$

By means of the thin-layer BWM, one find that only the integrals I_1 and I_2 contribute to the leading and subleading terms to the entropy.

Now consider the integrals I_1 and I_2.

$$\beta_h^{-1} = \frac{\kappa_h}{2\pi} = \frac{\Delta'_{r_h}}{\chi^2 A_h},$$
$$A_h = 4\pi(r_h^2 + a^2)/\chi\,. \qquad (29)$$

Take the angular velocity of a quantum field in the thin layer near the horizon of Kerr-de Sitter black hole as $\Omega_h = a/(r_h^2 + a^2)$, and expand Δ_r close to r_h as

$$\Delta_r = \Delta'_{r_h}(r - r_h) + \frac{1}{2}\Delta''_{r_h}(r - r_h)^2 + \cdots, \qquad (30)$$

and then expand three quantities $\tilde{g}_{tt}$, P, and W in terms of the surface gravity

$$\kappa_h = \Delta'_{r_h}/[2\chi(r_h^2 + a^2)] = 2\pi\Delta'_{r_h}/(\chi^2 A_h)$$

as follows

$$\begin{aligned}\tilde{g}_{tt} &= \frac{\Delta_\theta a^2 \sin^2\theta (r^2 - r_h^2)^2 - \Delta_r \Sigma_h^2}{\chi^2 (r_h^2 + a^2)^2 \Sigma} \\ &\approx \frac{-2\kappa_h \Sigma_h (r - r_h)}{\chi(r_h^2 + a^2)}\Big[1 - \Big(\frac{2r_h}{\Sigma_h} - \frac{\Delta''_{r_h}}{2\Delta'_{r_h}} \\ &\qquad + \frac{4r_h^2 \Delta_\theta a^2 \sin^2\theta}{\Delta'_{r_h}\Sigma_h^2}\Big)(r - r_h)\Big] + \cdots,\end{aligned}$$

$$
\begin{aligned}
P &= \frac{4s^2+2}{l^2} + \frac{4s^2a^2\cos^2\theta}{\Sigma^3}\Big[\Delta_r \\
&\quad -\Delta_\theta(r^2+a^2) + \frac{a^2\sin^2\theta}{l^2}\Sigma\Big] \\
&\approx \frac{4s^2+2}{l^2} + \frac{4s^2a^2\cos^2\theta}{\Sigma_h^3}\Big[\frac{a^2\sin^2\theta}{l^2}\Sigma_h \\
&\quad -\Delta_\theta(r_h^2+a^2)\Big] + \cdots, \\
W &= \frac{-a\cos\theta}{\chi(r_h^2+a^2)\Sigma^2}\Big\{\Big[\chi\Sigma + 2a^2\sin^2\theta \\
&\quad \times(1-\frac{r^2}{l^2})\Big](r^2-r_h^2) + 2\Delta_r\Sigma_h\Big\} \\
&\approx \frac{-4\kappa_h a\cos\theta}{\Sigma_h}\Big\{1 + \frac{r_h}{\Delta'_{r_h}}\Big[\chi + \frac{2a^2\sin^2\theta}{\Sigma_h} \\
&\quad \times(1-\frac{r_h^2}{l^2})\Big]\Big\}(r-r_h) + \cdots.
\end{aligned} \tag{31}
$$

where $\Sigma_h = r_h^2 + a^2\cos^2\theta$.

Expanding the integrands in $I_1 \sim I_2$ as

$$\frac{\sqrt{-g}}{\tilde{g}_{tt}^2} \approx \frac{(r_h^2+a^2)^2 \sin\theta}{4\kappa_h^2 \Sigma_h}\Big[\frac{1}{(r-r_h)^2} + \Big(\frac{6r_h}{\Sigma_h} - \frac{\Delta''_{r_h}}{\Delta'_{r_h}} + \frac{8r_h^2\Delta_\theta a^2 \sin^2\theta}{\Delta'_{r_h}\Sigma_h^2}\Big)\frac{1}{r-r_h}\Big] + \cdots$$

$$\approx \frac{(r_h^2+a^2)\sin\theta}{4\chi\kappa_h^3\Sigma_h}\Big[\frac{\Delta'_{r_h}}{2(r-r_h)^2} + \Big(\frac{3r_h\Delta'_{r_h}}{\Sigma_h} - \frac{1}{2}\Delta''_{r_h} + \frac{4r_h^2\Delta_\theta a^2\sin^2\theta}{\Sigma_h^2}\Big)\frac{1}{r-r_h}\Big] + \cdots ,$$

$$\frac{\sqrt{-g}}{\tilde{g}_{tt}}P = \frac{-(r_h^2+a^2)\sin\theta}{2\chi\kappa_h(r-r_h)}\Big\{\frac{4s^2+2}{l^2} + 4s^2a^2\cos^2\theta\Big[\frac{a^2\sin^2\theta}{l^2\Sigma_h^2} - \frac{\Delta_\theta(r_h^2+a^2)}{\Sigma_h^3}\Big]\Big\} + \cdots , \qquad (32)$$

and carrying out the integrals with respect to θ and r, we finally arrive at

$$I_1 = \frac{2}{\chi\kappa_h^3}\Big[\frac{15(r_h^2+a^2)}{4\epsilon^2} + \Big(1 - \frac{3r_h^2+a^2}{2l^2}\Big)\ln\frac{\Lambda}{\epsilon}\Big],$$

$$I_2 = \frac{1}{2\chi\kappa_h}\Big\{-\frac{4(r_h^2+a^2)}{l^2} + s^2\Big[\frac{a^2-r_h^2}{r_h^2} + \frac{r_h^2-a^2}{l^2} + \Big(\frac{r_h^2+a^2}{r_h^2} - \frac{9r_h^2+a^2}{l^2}\Big) \times\frac{r_h^2+a^2}{ar_h}\arctan\Big(\frac{a}{r_h}\Big)\Big]\Big\}\ln\frac{\Lambda}{\epsilon}. \qquad (33)$$

where the ultraviolet cutoff $\varepsilon = \eta^2\Delta'_{r_h}/(4\Sigma_h)$ is replaced by the proper distance

$$\eta = \int_{r_h}^{r_h+\varepsilon}\sqrt{g_{rr}}dr \approx 2\Big(\frac{\varepsilon\Sigma}{\Delta'_{r_h}}\Big)^{1/2}.$$

The new ultraviolet cutoff ϵ and infrared cut-off Λ are defined by

$$\eta^2 = \frac{2\epsilon^2}{15}\frac{N-1}{N}\frac{\Sigma_h}{ar_h}\arctan\Big(\frac{a}{r_h}\Big) \approx \frac{2\epsilon^2}{15},$$
$$N = \Lambda^2/\epsilon^2. \qquad (34)$$

The statistical-mechanical entropy:

$$
\begin{aligned}
S/g_s &= \pi^3 \frac{15+(-1)^{2s}}{180\beta^3} I_1 + \pi \frac{3+(-1)^{2s}}{12\beta} I_2 \\
&= \frac{15+(-1)^{2s}}{90\chi(\beta\kappa_h/\pi)^3} \Big[\frac{15(r_h^2+a^2)}{4\epsilon^2} \\
&\quad + \Big(1 - \frac{3r_h^2+a^2}{2l^2}\Big) \ln\frac{\Lambda}{\epsilon} \Big] \\
&\quad + \frac{3+(-1)^{2s}}{24\chi(\beta\kappa_h/\pi)} \Big\{ -\frac{4(r_h^2+a^2)}{l^2} \\
&\quad + s^2 \Big[\frac{a^2-r_h^2}{r_h^2} + \frac{r_h^2-a^2}{l^2} \\
&\quad + \Big(\frac{r_h^2+a^2}{r_h^2} - \frac{9r_h^2+a^2}{l^2} \Big) \\
&\quad \times \frac{r_h^2+a^2}{ar_h} \arctan\Big(\frac{a}{r_h}\Big) \Big] \Big\} \ln\frac{\Lambda}{\epsilon} . \qquad (35)
\end{aligned}
$$

Assuming that the field is **in the Hartle-Hawking vacuum state** and taking $\beta = \beta_h$, we get that the entropy is given by

$$S/g_s = \frac{15 + (-1)^{2s}}{16}\Big[\frac{A_h}{48\pi\epsilon^2} + \frac{1}{45\chi} \times \Big(1 - \frac{3r_h^2 + a^2}{2l^2}\Big)\ln\frac{\Lambda}{\epsilon}\Big] + \frac{3 + (-1)^{2s}}{4} \times \Big\{-\frac{A_h}{12\pi l^2} + \frac{s^2}{12\chi}\Big[\frac{a^2 - r_h^2}{r_h^2} + \frac{r_h^2 - a^2}{l^2} + \Big(\frac{r_h^2 + a^2}{r_h^2} - \frac{9r_h^2 + a^2}{l^2}\Big)\frac{r_h^2 + a^2}{a r_h} \times \arctan\Big(\frac{a}{r_h}\Big)\Big]\Big\}\ln\frac{\Lambda}{\epsilon}. \quad (36)$$

V. DISCUSSIONS AND CONCLUSIONS

(a) The entropies given above have summed up the contribution from the maximal and minimal spin-weight states of a quantum field. The total entropy of the Kerr-de Sitter black hole is a linear sum of that of the two horizons.

The calculations here are valid both for the black hole event horizon case and for the cosmological horizon case. We think it is also valid for the black hole event horizon of the **Kerr-anti de Sitter** space by changing the sign of the cosmological constant.

(b) The entropies depend not only on the spins of the particles but also on the cosmological constant except different spin field obey different statistics. They rely on the quadratic terms of s^2 and $-1/l^2$ as well as a^2.

(c) Both the contribution of the spins and that of the cosmological constant to the entropies are in subleading order.

(d) The logarithmic term from the spins of the particles not only depend on the **spin-rotation coupling effect** but also on the coupling between the spins of particles and the cosmological constant.

(e) Two special cases may be very interesting:

I. Schwarzschild-de Sitter black hole Case:

$$S/g_s = \frac{15+(-1)^{2s}}{16}\Big[\frac{A_h}{48\pi\epsilon^2} + \frac{1}{45}\Big(1-\frac{3r_h^2}{2l^2}\Big)\ln\frac{\Lambda}{\epsilon}\Big] - \frac{3+(-1)^{2s}}{4}\cdot\frac{1+2s^2}{12\pi l^2}A_h \ln\frac{\Lambda}{\epsilon}, \tag{37}$$

where $A_h = 4\pi r_h^2$.

II. Kerr black hole Case:

$$S/g_s = \frac{15+(-1)^{2s}}{16}\Big(\frac{A_h}{48\pi\epsilon^2} + \frac{1}{45}\ln\frac{\Lambda}{\epsilon}\Big) + \frac{3+(-1)^{2s}}{4}\cdot\frac{s^2}{12}\Big[\frac{a^2-r_h^2}{r_h^2} + \frac{(r_h^2+a^2)^2}{ar_h^3}\arctan\Big(\frac{a}{r_h}\Big)\Big]\ln\frac{\Lambda}{\epsilon}, \tag{38}$$

where $A_h = 4\pi(r_h^2+a^2)$.

Thank you !