symptotic Structure of Einstein-Maxwell-Dilaton Theory and Its Five Dimensional Origin

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- In this framework, the gravitational radiation is characterized by the news functions and the mass of the system always decreases whenever news functions exist.
- This demonstrates that gravitational waves exist in the full Einstein theory rather as an artifact of linearization.

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- In the last several years, Strominger proposed triangle relation among asymptotic symmetry, soft theorems for graviton amplitudes and gravitational memory effects.

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- For extending Bondi's framework to include a matter coupled system with the same power series expansion, the matter fields are necessarily massless.
- The Einstein-Maxwell theory in Bondi gauge was studied in *[van der Burg, 1969][Bieri, Chen, Yau, 2011]*.
- However, the effect of other types of matter fields is less stressed in literatures.

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- In this work, we study asymptotic structure of 4d Einstein-Maxwell-Dilaton theory.
- Its uplift gives the asymptotic structure of 5d pure gravity where the topological of infinity is $S^2 \times S^1$ instead of S^3.

4d Einstein-Maxwell-Dilaton(EMD) Theory

- $4d$ EMD theory includes gravity, Maxwell field and massless scalar. The Lagrangain is

$$\mathcal{L} = \sqrt{-g}\left[R - \frac{1}{4}e^{a\varphi}F^2 - \frac{1}{2}(\partial\varphi)^2\right], \qquad F = dA. \tag{1}$$

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- When a is one of $0, \frac{1}{\sqrt{3}}, 1, \sqrt{3}$, the EMD theory can all be embedded in the $\mathcal{N} = 2$ STU supergravity , which is pure $\mathcal{N} = 2$ supergravity with three vector multiplets *[Duff, Liu, Rahmfeld, hep-th/9508094]*.

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- The $a = 0$ case can be reduced to Einstein-Maxwell theory which is the bosonic sector of $\mathcal{N} = 2$ supergravity.
- The $a = \sqrt{3}$ case can be Kaluza-Klein theory obtained from the circle reduction from pure gravity in five dimensions.

Equations of motion

- The dilaton, Maxwell and Einstein equations are

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) - \frac{a}{4}\sqrt{-g}e^{a\varphi}F^2 = 0, \qquad \partial_\nu(\sqrt{-g}e^{a\varphi}F^{\mu\nu}) = 0,$$

$$(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) - \frac{1}{2}e^{a\varphi}F_{\mu\rho}F_\nu{}^\rho + \frac{1}{8}g_{\mu\nu}e^{a\varphi}F^2 - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi$$
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$$+\frac{1}{4}g_{\mu\nu}(\partial\varphi)^2 = 0. \qquad (2)$$

- The Einstein equation is equivalent to

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}e^{a\varphi}F_{\mu\rho}F_\nu{}^\rho + \frac{1}{8}g_{\mu\nu}e^{a\varphi}F^2 - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi = 0, \qquad (3)$$

Bondi gauge

- We study the above EMD theory in four dimensions in Bondi gauge. The metric has the form *[Bondi, van der Burg, Metzner, 1962]*

$$
\begin{aligned}
ds^2 = &\left[-\frac{V(u,r,\theta)}{r}e^{2\beta(u,r,\theta)} + U(u,r,\theta)^2 r^2 e^{2\gamma(u,r,\theta)}\right] du^2 \\
&- 2e^{2\beta(u,r,\theta)}dudr - 2U(u,r,\theta)r^2 e^{2\gamma(u,r,\theta)}dud\theta \\
&+ r^2\left[e^{2\gamma(u,r,\theta)}d\theta^2 + e^{-2\gamma(u,r,\theta)}\sin^2\theta d\phi^2\right]. \qquad (4)
\end{aligned}
$$

$$
A = A_u(u,r,\theta)du + A_\theta(u,r,\theta)d\theta. \qquad (5)
$$

Bondi gauge

- The inverse metric is simple

$$g^{\mu\nu} = \begin{pmatrix} 0 & -e^{-2\beta} & 0 & 0 \\ -e^{-2\beta} & \frac{V}{r}e^{-2\beta} & -Ue^{-2\beta} & 0 \\ 0 & -Ue^{-2\beta} & \frac{e^{-2\gamma}}{r^2} & 0 \\ 0 & 0 & 0 & \frac{e^{2\gamma}}{\sin^2\theta r^2} \end{pmatrix}. \quad (6)$$

Boundary conditions

- The falloff conditions for the functions β, γ, U, V are

$$\beta = \mathcal{O}(r^{-1}), \quad \gamma = \mathcal{O}(r^{-1}), \quad U = \mathcal{O}(r^{-2}), \quad V = \mathcal{O}(r). \tag{7}$$

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- The boundary conditions for gauge fields are

$$A_u = \mathcal{O}(r^{-1}), \quad A_\theta = \mathcal{O}(1), \quad \varphi = \mathcal{O}(r^{-1}). \tag{8}$$

Constraints among equations of motion

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- The constraints among them are the following identities

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- Making use of these constraints, we are able to arrange the fifteen equations of motion into four classes.

Arranging the equations of motion

- Class 1: five hypersurface equations:

$$\begin{aligned} &\partial_\nu(\sqrt{-g}e^{a\varphi}F^{u\nu}) = 0, \\ &E_{rr} = E_{r\theta} = E_{r\phi} = 0, \\ &E_{\theta\theta}g^{\theta\theta} + E_{\phi\phi}g^{\phi\phi} = 0. \end{aligned} \tag{10}$$

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$$E_{\theta\theta}g^{\theta\theta} + E_{\phi\phi}g^{\phi\phi} = 0.$$

- Class 2: five standard equations:

$$\partial_\nu(\sqrt{-g}e^{a\varphi}F^{\theta\nu}) = \partial_\nu(\sqrt{-g}e^{a\varphi}F^{\phi\nu}) = 0,$$
$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) - \frac{a}{4}\sqrt{-g}e^{a\varphi}F^2 = 0, \tag{11}$$
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- Once the hypersurface equations and standard equations are satisfied, the(constraint) identities (9) yield that the trivial equation is satisfied automatically
- and the supplementary equations are left with only one order in the $\frac{1}{r}$ expansions.

Hypersurface equations

- $$\partial_r \beta = \frac{r}{2}(\partial_r \gamma)^2 + \frac{r}{8}(\partial_r \varphi)^2 + \frac{1}{8r} e^{a\varphi - 2\gamma}(\partial_r A_\theta)^2. \tag{14}$$

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- $$\partial_r L = \frac{1}{\sin\theta}\partial_\theta\left[\sin\theta e^{a\varphi-2\gamma}\partial_r A_\theta\right], \qquad (16)$$
- $$\begin{aligned}\partial_r\left[r^4 e^{2(\gamma-\beta)}\partial_r U\right] &= 2r^2\Big[\partial_r\partial_\theta(\beta-\gamma) + 2\partial_r\gamma\partial_\theta\gamma - \frac{2\partial_\theta\beta}{r} \\ &\quad -2\partial_r\gamma\cot\theta\Big] + r^2\partial_r\varphi\partial_\theta\varphi + L\partial_r A_\theta. \qquad (17)\end{aligned}$$

Hypersurface equations

- $$\begin{aligned}
\partial_r V &= 2r\partial_\theta U + \frac{1}{2}r^2\partial_r\partial_\theta U - \frac{1}{4}r^4 e^{2(\gamma-\beta)}(\partial_r U)^2 + \frac{1}{2}r^2\partial_r U\cot\theta \\
&+ 2rU\cot\theta + e^{2(\beta-\gamma)}\Big[1 - (\partial_\theta\beta)^2 - \partial_\theta\beta\cot\theta + 2\partial_\theta\beta\partial_\theta\gamma \\
&+ 3\partial_\theta\gamma\cot\theta - 2(\partial_\theta\gamma)^2 - \partial_\theta^2\beta + \partial_\theta^2\gamma\Big] \\
&- \frac{1}{4r^2}L^2 e^{2\beta-a\varphi} - \frac{1}{4}e^{2(\beta-\gamma)}(\partial_\theta\varphi)^2 \qquad (18)
\end{aligned}$$

Hypersurface equations

- β is fixed by $\gamma, \varphi, A_\theta$.
  U is fixed by $\beta, \gamma, \varphi, A_\theta$.
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  V is fixed by $\beta, \gamma, U, \varphi, A_\theta, A_u$.
- Initial data: $\gamma, \varphi, A_\theta$.
- *Their time evolutions (w. r. t. u) are controlled by the standard equations.*

Standard equations

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- The rest three equations,

$$\frac{1}{2}re^{2\beta}E_{\phi\phi}g^{\phi\phi} = 0, \tag{19}$$

$$\frac{1}{2\sin\theta}e^{2\gamma-a\varphi}\partial_\nu(\sqrt{-g}e^{a\varphi}F^{\theta\nu}) = 0, \tag{20}$$

$$\frac{1}{2r\sin\theta}\left[\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) - \frac{a}{4}\sqrt{-g}e^{a\varphi}F^2\right] = 0, \tag{21}$$

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  will determine the time evolution of γ, φ and A_θ, which will be calculated in this subsection.
- *There is no constraint at the order $\mathcal{O}(\frac{1}{r})$ of γ and φ, and at the order $\mathcal{O}(1)$ of A_θ from those three equations. They are related to the "news" functions in the system which indicating radiations.*

Solution space in series expansion

- Suppose that γ, φ and A_θ are given in $\frac{1}{r}$ series expansion as initial data

$$\gamma = \frac{c(u,\theta)}{r} + \sum_{a=3}^{\infty} \frac{\gamma_a(u,\theta)}{r^a}, \tag{22}$$

$$\varphi = \sum_{a=1}^{\infty} \frac{\varphi_a(u,\theta)}{r^a}. \tag{23}$$

$$A_\theta = \mathcal{A}_0(u,\theta) + \sum_{a=1}^{\infty} \frac{\mathcal{A}_a(u,\theta)}{r^a}, \tag{24}$$

Solution space in series expansion

- Then hypersurface equations gives

$$\beta = -\frac{4c^2+\varphi_1^2}{16r^2} - \frac{\varphi_1\varphi_2}{6r^3} - \frac{12c\gamma_3 + 2\varphi_2^2 + 3\varphi_1\varphi_3 + \frac{1}{2}\mathcal{A}}{16r^4} + \mathcal{O}(r^{-5}), \tag{25}$$

$$U = -\frac{\partial_\theta c + 2c\cot\theta}{r^2} + \frac{4c\left(\partial_\theta c + 2c\cot\theta\right) - N(u,\theta)}{3r^3} + \mathcal{O}(r^{-4}), \tag{26}$$

$$A_u = -\frac{q(u,\theta)}{r} - \frac{\mathcal{A}_1\cot\theta + \partial_\theta\mathcal{A}_1 - aq\varphi_1}{2r^2} + \mathcal{O}(r^{-3}), \tag{27}$$

and

$$V = r - M(u,\theta) + \mathcal{O}(r^{-1}), \tag{28}$$

where $M(u,\theta)$, $N(u,\theta)$ and $q(u,\theta)$ are the integration "constants" from solving the partial differential equation associate with r.

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- In particular the first order of the standard equations are listed as follows:

$$\partial_u \varphi_2 = -\frac{1}{2}(\partial_\theta^2 \varphi_1 + \cot\theta \partial_\theta \varphi_1). \tag{29}$$

$$\partial_u \mathcal{A}_1 = c\partial_u \mathcal{A}_0 - \frac{1}{2}\partial_\theta q - \frac{1}{2} a\varphi_1 \partial_u \mathcal{A}_0. \tag{30}$$

$$\begin{aligned}\partial_u \gamma_3 = \frac{1}{96}\Big[& c^2(16 - 32\csc^2\theta) - 4\cot\theta N - 3\cot\theta \partial_\theta \varphi_1 \varphi_1 - 3(\partial_\theta \varphi_1)^2 \\ & + 3\varphi_1 \partial_\theta^2 \varphi_1 + 4c\left(6M + 3\cot\theta \partial_\theta c + 5\partial_\theta^2 c\right) + 4\left(\partial_\theta N + 5(\partial_\theta c)^2\right. \\ & \left. -3\mathcal{A}_1 \partial_u \mathcal{A}_0\right)\Big]. \end{aligned} \tag{31}$$

News functions

- All the time evolution equations of the sub-leading terms in γ, φ and A_θ can be derived recursively order by order. However the time evolution of c, $\mathcal{A}_0$ and φ_1 are not constrained. Hence, $\dot{c}$, $\dot{\mathcal{A}}_0$ and $\dot{\varphi}_1$[1] are the news functions of this system that indicate gravitational, electromagnetic, and scalar radiations.

[1] An overdot denotes a time derivative ∂_u.

Supplementary equations

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- Equation $E_{u\phi} = 0$ holds automatically, again from the assumption that the system is ϕ-independent.
- The rest three supplementary equations determine the time evolution of the integration constants M, N and q.

Conservation of the electric charges

- From $\partial_\nu(\sqrt{-g}e^{a\varphi}F^{r\nu}) = 0$, we obtain

$$\partial_u q = -\cot\theta \partial_u \mathcal{A}_0 - \partial_u \partial_\theta \mathcal{A}_0. \tag{32}$$

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- From the identity

$$\oint \sin\theta(\cot\theta\partial_u\mathcal{A}_0 + \partial_u\partial_\theta\mathcal{A}_0)d\theta d\phi = 2\pi\partial_u\mathcal{A}_0\sin\theta\ |_0^\pi = 0, \tag{33}$$

we can conclude that the total electric charge Q, defined by

$$Q = \oint q(u,\theta)\sin\theta d\theta d\phi, \tag{34}$$

is conserved. This is not surprising because the dilaton scalar field is real and it cannot carry electric charges.

Mass-loss formula

- The supplementary equation $E_{uu} = 0$ leads to

$$\partial_u M = -2(\dot{c})^2 - \frac{1}{2}(\dot{\mathcal{A}}_0)^2 - \frac{1}{2}(\dot{\varphi}_1)^2 + 3\cot\theta \partial_u \partial_\theta c + \partial_u \partial_\theta^2 c - 2\partial_u c. \quad (35)$$

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  This is the generalized Bondi mass-loss formula in the four dimensional EMD theory.

- We define the mass density

$$m = M - \frac{1}{\sin\theta}\partial_\theta\left(2\cos\theta c + \sin\theta\partial_\theta c\right). \quad (36)$$

  Inserting the mass density into the generalized Bondi mass-loss formula (35), one obtains

$$\partial_u m = -2(\dot{c})^2 - \frac{1}{2}(\dot{\mathcal{A}}_0)^2 - \frac{1}{2}(\dot{\varphi}_1)^2. \quad (37)$$

Mass-loss formula

- Thus, we have the following theorem in four dimensional Einstein-Maxwell-dilaton theory:
  The mass density at any angle of the system can never increase. It is a constant if and only if there is no news.

Asymptotic symmetries

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- The generating infinitesimal transformations are given by

$$\delta_{(\xi,\chi)} g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}, \quad \delta_{(\xi,\chi)} A_\mu = \partial_\mu \chi + \mathcal{L}_\xi A_\mu, \quad \delta_{(\xi,\chi)} \varphi = \mathcal{L}_\xi \varphi. \quad (38)$$

Asymptotic symmetries

- The complete set of local symmetry involves a pair (ξ, χ) of a vector field $\xi = \xi^\mu \partial_\mu$ and an internal gauge parameter χ.
- The generating infinitesimal transformations are given by

$$\delta_{(\xi,\chi)} g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}, \quad \delta_{(\xi,\chi)} A_\mu = \partial_\mu \chi + \mathcal{L}_\xi A_\mu, \quad \delta_{(\xi,\chi)} \varphi = \mathcal{L}_\xi \varphi. \quad (38)$$

- The infinitesimal transformation parameters are independent of ϕ in order to keep the ϕ-independence of the fields.

Asymptotic symmetries

- Gauge conditions

$$g_{rr} = g_{r\phi} = g_{r\theta} = g_{u\phi} = g_{\theta\phi} = A_r = 0, \tag{39}$$

lead to

$$\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{r\phi} = \mathcal{L}_\xi g_{r\theta} = \mathcal{L}_\xi g_{u\phi} = 0 \mathcal{L}_\xi g_{\theta\phi} = 0, \delta_{(\xi,\chi)} A_r = 0. \tag{40}$$

We have one more gauge condition from angular part of metric elements

$$\mathcal{L}_\xi \left(\frac{g_{\phi\phi}}{g^{\theta\theta}}\right) = 0. \tag{41}$$

Asymptotic symmetries

- The transformation should also respect the boundary conditions

$$\beta = \mathcal{O}(r^{-1}), \quad \gamma = \mathcal{O}(r^{-1}), \quad U = \mathcal{O}(r^{-2}), \quad V = \mathcal{O}(r). \tag{42}$$

$$A_u = \mathcal{O}(r^{-1}), \quad A_\theta = \mathcal{O}(1), \quad \varphi = \mathcal{O}(r^{-1}). \tag{43}$$

Asymptotic symmetries

- The results

$$
\begin{aligned}
\xi^u &= f = T + uy\cos\theta, \\
\xi^r &= -\frac{r}{2}\left(\partial_\theta \xi^\theta + \cot\theta\,\xi^\theta - g^{r\theta} g_{ur}\partial_\theta f\right), \\
\xi^\theta &= y\sin\theta + \partial_\theta f \int_r^\infty dr\; g_{ru} g^{\theta\theta}, \\
\xi^\phi &= \xi^\phi,
\end{aligned} \tag{44}
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\chi = \epsilon - \int_r^\infty dr\; A_\theta g^{\theta\theta} g_{ru} \partial_\theta f. \tag{45}
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- Notice that ξ^r, ξ^θ, χ depend on the coupling constant a through their dependence on the metric and Maxwell field.

Asymptotic symmetry algebra

- The asymptotic symmetry transformations satisfy a modified algebra introduced in *[Barnich etal, 2001][Barnich etal, 2013]*

$$[(\xi_1, \chi_1), (\xi_2, \chi_2)]_M = (\hat{\xi}, \hat{\chi}), \tag{46}$$

where

$$\hat{\xi} = [\xi_1, \xi_2] - \delta_{(\xi_1,\chi_1)}\xi_2 + \delta_{(\xi_2,\chi_2)}\xi_1, \tag{47}$$

$$\hat{\chi} = \xi_1^\mu \partial_\mu \chi_2 - \xi_2^\mu \partial_\mu \chi_1 - \delta_{(\xi_1,\chi_1)}\chi_2 + \delta_{(\xi_2,\chi_2)}\chi_1 \tag{48}$$

Asymptotic symmetry algebra

- The algebra is closed which can be seen from straightforward computation

$$\begin{aligned}
&\hat{\xi}^u = \hat{f} = y_1 \sin\theta(\partial_\theta T_2 - \cot\theta T_2) - y_2 \sin\theta(\partial_\theta T_1 - \cot\theta T_1), \\
&\partial_r(\hat{\xi}^\theta) = -g_{ur} g^{\theta\theta} \partial_\theta \hat{f}, \\
&\partial_r(\frac{\hat{\xi}^r}{r}) = \frac{1}{2}\left[\partial_\theta(g^{\theta\theta} g_{ur} \partial_\theta \hat{f}) + \cot\theta(g^{\theta\theta} g_{ur} \partial_\theta \hat{f}) + \partial_r(g^{r\theta} g_{ur} \partial_\theta \hat{f})\right], \\
&\partial_r(\hat{\chi}) = A_\theta g^{\theta\theta} g_{ru} \partial_\theta \hat{f}.
\end{aligned} \tag{49}$$

- When $r \to \infty$, the algebra is reduced to

$$[(T_1, y_1, \epsilon_1), (T_2, y_2, \epsilon_2)] = (\hat{T}, \hat{y}, \hat{\epsilon}), \tag{50}$$

where

$$\hat{T} = y_1 \sin\theta(\partial_\theta T_2 - \cot\theta T_2) - (1 \leftrightarrow 2), \tag{51}$$

$$\hat{y} = 0, \hat{\epsilon} = y_1 \sin\theta \partial_\theta \epsilon_2 - (1 \leftrightarrow 2). \tag{52}$$

Mode expansion

- To implement mode expansions, we define $t = \tan\frac{\theta}{2}$. In the new coordinate, we have

$$\hat{T} = y_1(t\partial_t T_2 - \frac{1-t^2}{1+t^2}T_2) - (1 \leftrightarrow 2), \tag{53}$$

$$\hat{y} = 0, \tag{54}$$

$$\hat{\epsilon} = y_1 t\partial_t \epsilon_2 - (1 \leftrightarrow 2). \tag{55}$$

The basis vectors are chosen as

$$T_m = \left(\frac{t}{1+t^2}\right) t^m \partial_u, \quad Y_0 = t\partial_t, \quad \epsilon_m = t^m. \tag{56}$$

In terms of the basis vector, the asymptotic symmetry algebra is

$$[T_m, T_n] = [\epsilon_m, \epsilon_n] = [T_m, \epsilon_n] = 0, \tag{57}$$

$$[Y_0, T_n] = nT_n, \quad [Y_0, \epsilon_n] = n\epsilon_n. \tag{58}$$

Uplifting to five dimensions

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$$ds_5^2 = e^{-\frac{1}{\sqrt{3}}\varphi} ds_4^2 + e^{\frac{2}{\sqrt{3}}\varphi}(dz + A_\mu dx^\mu)^2. \tag{59}$$

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- The extra news functions arises because the asymptotic spacetimes in five dimensions is a product of four-dimensional Minkowski spacetimes and a circle.

5d Asymptotic symmetries

- 5d gauge conditions are

$$g_{rr} = g_{r\theta} = g_{r\phi} = g_{rz} = g_{u\phi} = g_{\theta\phi} = g_{\phi z} = 0, \qquad (60)$$

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- The asymptotic Killing vector ξ^μ is independent of ϕ, z.
- 5d boundary conditions:

$$g_{ur} = -1+\mathcal{O}(r^{-1}), \quad g_{u\theta} = \mathcal{O}(1), \quad g_{uz} = \mathcal{O}(r^{-1}), \quad g_{\theta\theta} = r^2+\mathcal{O}(r). \quad (62)$$

5d Asymptotic Killing vectors

- The solutions are

$$\begin{aligned}
\xi^u &= f = T + \frac{1}{2}(\partial_\theta Y^\theta + \cot\theta Y^\theta)u, \\
\xi^r &= -\frac{r}{2}\left(\partial_\theta \xi^\theta + \cot\theta \xi^\theta - g^{r\theta} g_{ur} \partial_\theta f\right), \\
\xi^\theta &= Y + \partial_\theta f \int_r^\infty dr\ g_{ru} g^{\theta\theta}, \\
\xi^\phi &= \xi^\phi, \\
\xi^z &= \epsilon + \partial_\theta f \int_r^\infty dr\ g_{ru} g^{z\theta}.
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5d Asymptotic symmetry algebra

- The 5d asymptotic Killing vectors satisfy the following algebra

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Unsurprisingly, we recover the same algebra as the one in 4d EMD theory.

Conclusions

- We investigate the asymptotics in cases with coupled massless dynamical fields with various spins.

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- This is a case study of the asymptotics of five dimensional pure gravity among a well-chosen class of solutions avoiding half-integer powers in $1/r$ expansions.
- Asymptotic symmetry algebras in both four and five dimensional cases were computed and they are the same.

Future directions

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- It will be of interest to see whether the asymptotic behavior has strong dependence on the chosen null direction.
- We may need to study behavior of four dimensional fields at timelike infinity in additional to the behavior at null infinity studied here.

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- This work and our studies on asymptotics of four dimensional EMD theory here also strongly motivates us to study triangular equivalent relations of *Strominger* among asymptotic symmetries, various soft theorems and memory effects in this theory.

Thanks for Your Attention !