Holographic Complexity Bounds

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Outline

- Introduction
- We we can learn from the Schwarzschild-AdS black hole?
- Reissner-Nordström-AdS black hole
- General statements, Complexity and volume of black hole singularity
- Further tests
- Negative volume and violation of Lloyd's bound
- Summary and Conclusions

Introduction

- Holography principle provides a new powerful methodology to study the dynamics of various strongly coupled CFTs and CMTs. [Maldacena, ...]

- Typical technique is to consider a bulk background that is asymptotic to AdS and study its perturbations and derive the properties of the dual field theory, using the holographic dictionary.

- Holography is particularly useful to give some universal structures of the CFTs.

- Schwarzschild-AdS black hole is unique in this regard since it does not depend on matter; it can either give a universal result or a bound for the general systems.

- Holography was proposed to study quantum information; two conjectures on quantum computational complexity: CV or CA conjectures. [Susskind, et al.]

Lloyd's bound

There is an upper bound of the growth rate of the quantum complexity [Lloyd quant-ph/9908043]

$$\dot{\mathcal{C}} \leq 2M\,.$$

It depends on the mass of the system only!

What object in our universe is specified by the mass only? **The Schwarzschild black hole!**

It is thus natural to conjecture that the black hole is the fastest scrambler.

The process of the black hole formation is a process of scrambling and the later time growth rate is $2M$, i.e. twice of the black hole mass.

How to quantify this quantity from black hole physics?

The CV conjecture

CV conjecture: "complex=black hole volume" [Susskind 1402.5674; Stanford, Susskind 1406.2678]

Space volume involving singularity is notoriously difficult to define.

The trouble of this conjecture is that $\dot{\mathcal{C}}$ and V have different dimensions and requires ambiguous factor.

The cosmological constant of AdS black hole can be viewed as positive pressure and can thus introduce thermodynamcal volume

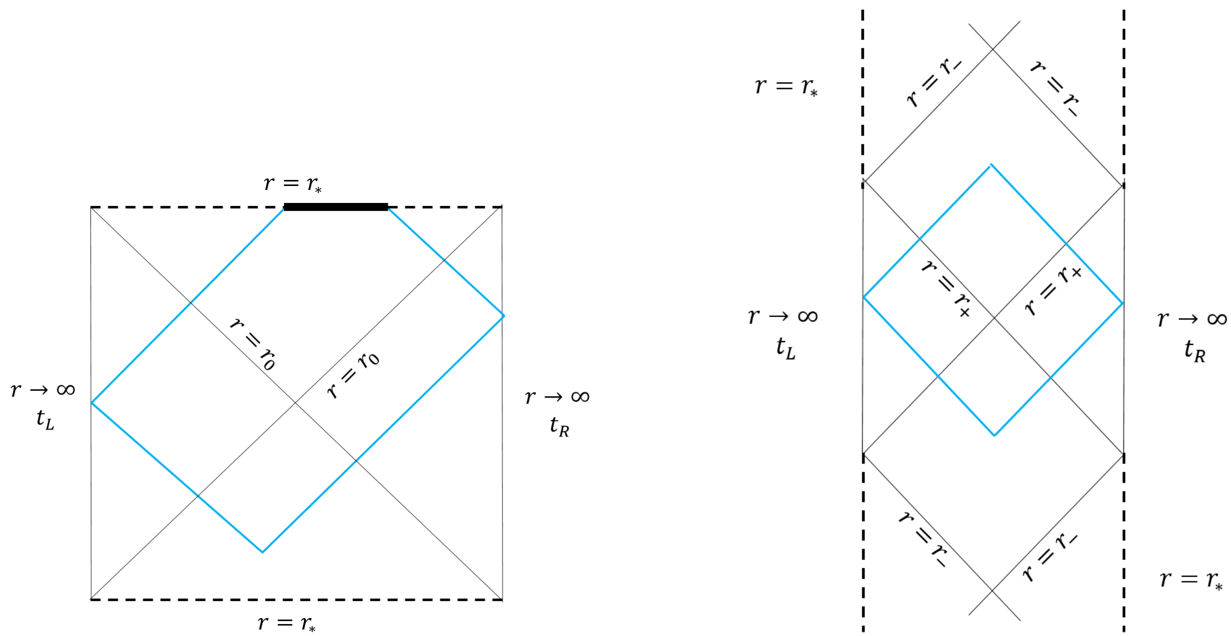
$$dM = TdS + V_{\text{th}} dP_{\text{th}} \,.$$

This leads to CV 2.0: [Couch, Fischler, Nguyen 1610.02038]

$$\dot{\mathcal{C}} \sim P_{\text{th}} V_{\text{th}} \,.$$

The CA conjecture

CA: "Complexity = Action" in Wheeler-De Witt patch.



For the Schwarzschild-AdS black hole, it is precisely

$$\dot{I}_{\mathrm{WDW}} = 2M\,.$$

Brown, Roberts, Susskind, Swingle, Zhao 1509.07876;1512.04993

Action in WDW patch: Lehner, Myers, Poisson, Sorkin 1609.00207

Questions to address

- Are the conjectures reasonable?
- Will Lloyd's bound be violated?
- Any relation between the CA and CV2.0 conjectures?
- What more can we learn?

Both CA and CV2.0 conjectures are now expressed in terms of black hole thermodynamical variables.

We can use the old technique to study the new phenomenon.

Schwarzschild-AdS black hole

The solution is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{2,k}\,, \qquad f = g^2 r^2 + k - \frac{2M}{r}\,,$$

with $k = 1, 0, -1$. The mass is M and the horizon is given by $f(r_+) = 0$, with

$$T = \frac{f'(r_+)}{4\pi}\,, \quad S = \pi r_+^2\,, \quad P_{\rm th} = \frac{3g^2}{8\pi}\,, \quad V_{\rm th} = \frac{4\pi}{3} r_+^3\,,$$

satisfying $dM = TdS + V_{\rm th} dP_{\rm th}$. Kastor,Ray, Traschen 0904.2765;Cvetic, Gibbons, Kubiznak,Pope,1012.2888

$$2M - 2P_{\rm th}V_{\rm th} = k r_+ \quad \rightarrow \quad \begin{cases} > 0, & k = 1; \\ = 0, & k = 0; \\ < 0, & k = -1. \end{cases}$$

Thus CA=CV2.0 for $k = 0$. Furthermore

$$\frac{1}{T}\frac{\partial(2M)}{\partial S} = 2\,.$$

i.e. $\dot{\mathcal{C}} \sim 2ST$. Can this generalize?

RN-AdS black hole

The solution is given by

$$f = g^2 r^2 + k - \frac{2M}{r} + \frac{Q^2}{r^2}\,, \qquad A = \frac{Q}{r} dt\,.$$

In addition to $(M, T, S, P_{\text{th}}, V_{\text{th}}$, we have $(Q, \Phi = Q/r_0)$ where r_0 denotes both $r_\pm$. The first law $dM = TdS + \Phi dQ + V_{\text{th}} dP_{\text{th}}$ is satisfied by both inner and outer horizons.

The action growth rate in the WDW patch is

$$\dot{I}_{\text{WDW}} = \Phi Q \Big|_{r_+}^{r_-}\,.$$

We find

$$\dot{I}_{\text{WDW}} = 2P_{\text{th}} \Delta V_{\text{th}} + k(r_+ - r_-)\,. \quad \Delta V_{\text{th}} = V_{\text{th}} \Big|_{r_-}^{r_+}\,.$$

Furthermore

$$\dot{I}_{\text{WDW}} - 2M = -\Big(2k + g^2 r_-^2 \frac{\eta^3 + \eta - 2)}{\eta - 1}\Big) r_-\,, \qquad \eta = \frac{r_+}{r_-}\,.$$

Thus $\dot{I}_{\text{WDW}} \le 2M$ for $k = 0, 1$, but not necessarily for $k = -1$.

Combined inequalities

$$
\begin{aligned}
k = 0,1: &\qquad 2P_{\mathrm{th}}\,\Delta V_{\mathrm{th}} \le \dot{I}_{\mathrm{WDW}} \le 2M\,,\\
k = -1: &\qquad 0 \le \dot{I}_{\mathrm{WDW}} \le 2P_{\mathrm{th}}\,\Delta V_{\mathrm{th}}\,.
\end{aligned}
$$

Differentiation

$$
\frac{1}{T}\frac{\partial \dot{I}_{\mathrm{WDW}}}{\partial S}\bigg|_{Q,P_{\mathrm{th}}} = 2 + \frac{X}{Y},
$$

$$
\begin{aligned}
X &= 2r_-\Big(g^4\left(6r_-^4 + 11r_+r_-^3 + 15r_+^2r_-^2 + 3r_+^3r_- + r_+^4\right)\\
&\qquad + g^2k\left(8r_-^2 + 5r_+r_- + 5r_+^2\right) + 2k^2\Big),\\
Y &= \left(r_+ - r_-\right)\left(g^2\left(3r_-^2 + 2r_+r_- + r_+^2\right) + k\right)\\
&\quad \times \left(g^2\left(r_-^2 + 2r_+r_- + 3r_+^2\right) + k\right).
\end{aligned}
$$

$X/Y \ge 0$ for $k = 0,1$. Similarly $\frac{1}{T}\frac{\partial(2P_{\mathrm{th}}\Delta V_{\mathrm{th}})}{\partial S}\Big|_{Q,P_{\mathrm{th}}} > 0$ for $k = 1$.

General Statements

$$\begin{aligned} k = 0, 1: \qquad & 2P_{\text{th}}\,\Delta V_{\text{th}} \le \dot{I}_{\text{WDW}} \le 2M, \\ k = -1: \qquad & 0 \le \dot{I}_{\text{WDW}} \le 2P_{\text{th}}\,\Delta V_{\text{th}}. \end{aligned}$$

$$\dot{I}_{\text{WDW}} = 2P_{\text{th}}\Delta V_{\text{th}}\,, \qquad k = 0\,.$$

We include in the above the Lloyd's bound even though it could be violated, since we would like to know the underlying reason for the violation.

We also propose

$$\frac{1}{T}\frac{\partial \dot{I}_{\text{WDW}}}{\partial S}\Big|_{Q,P_{\text{th}}} > C\,, \qquad k = 0, 1.$$

$$\frac{1}{T}\frac{\partial (2P_{\text{th}}\Delta V_{\text{th}})}{\partial S}\Big|_{Q,P_{\text{th}}} > 0\,, \qquad k = 1.$$

Will these proposal withstand the tests of all the black hole examples?

Black hole with two horizons

In this case, Huang,Feng, H. Lü 1611.02321]

$$\dot{I}_{\rm WDW} = \mathcal{H}\Big|_{-}^{+} = \Phi_i Q_i\Big|_{+}^{-}, \qquad \Delta V_{\rm th} = V_{\rm th}\Big|_{-}^{+}.$$

We can prove that CA=CV for $k=0$. The Smarr relation:

$$(D-3)M = (D-2)TS + (D-3)\Phi_i Q_i - 2P_{\rm th}V_{\rm th}.$$

The generalized Smarr relation for $k=0$ Liu, Lü, Pope 1507.02294]

$$(D-1)M = (D-2)(TS + \Phi_i Q_i).$$

Thus we have

$$2M - \Phi_i Q_i = 2P_{\rm th}V_{\rm th}, \qquad k=0.$$

Thus

$$\dot{I}_{\rm WDW} = 2P_{\rm th}\,\Delta V_{\rm th}.$$

We do not have a proof for any other relations, but we can test them with a variety of black hole examples.

Black hole with single horizon

Naively, we simply assume that $\dot{\mathcal{C}}_V = 2P_{\text{th}}V_{\text{th}}$. This will not always work since there are black holes with $V_{\text{th}} \neq 0$ even though it shrinks to cease to become a black hole.

We introduce a concept of the volume of black hole singularity V_{th}^0 and then

$$\Delta V_{\text{th}} = V_{\text{th}}^+ - V_{\text{th}}^0 \,.$$

We find then for $k = 0$

$$\dot{I}_{\text{WDW}} = 2M - \Phi_i Q_i - 2P_{\text{th}}V_{\text{th}}^0 = 2P_{\text{th}}\Delta V_{\text{th}} \,.$$

In the second equality, we made use of the Smarr relations.

For general k,

$$\dot{I}_{\text{WDW}} = 2M - \Phi_i Q_i - 2P_{\text{th}}V_{\text{th}}^0 + ku(r_+) \,,$$

where $u(r_+) \geq 0$ is a simple function of r_+.

What is the volume of singularity?

Volume of Black hole Singularity

The entropy or 1/4 of the area of horizon is a quantity defined on the horizon, but the area of a surface, can be define generally.

Can one generalize the thermodynamical volume V_{th} defined on the horizon to a general $V(r)$? The first effort [Feng, Lü 1701.05204]

The Schwarzschild and RN black hole:

$$V_{\text{th}} = \frac{4\pi}{3} r_+^3 \quad \rightarrow \quad V_{\text{th}}(r) = \frac{4\pi}{3} r^3 \,, \qquad V_{\text{th}}^0 = 0 \,.$$

Note that $V_{\text{th}}(r)$ is independent of g^2, or the pressure P_{th}, and hence the concept of volume can generalize to asymptotically-flat black holes.

The above generalization has an ambiguity since $g_{tt}(r_+) = 0$ and hence we can insert $g_{tt}(r)$ into $V_{\text{th}}(r)$.

We can remove this ambiguity by requiring that

$$\frac{\partial V_{\text{th}}}{\partial (g^2)} = 0.$$

Once we have $V_{\text{th}}(r)$, we can define $V_{\text{th}}^0 = V_{\text{th}}(0)$, which may not be zero.

Lloyd's bound and singularity of volume

For black holes with single horizons, we find

$$\dot{I}_{\mathrm{WDW}} = 2M - \Phi_i Q_i - 2P_{\mathrm{th}} V_{\mathrm{th}}^0 + k u(r_+)\,,$$

We find examples of $V_{\mathrm{th}}^0 > 0$, $V_{\mathrm{th}}^0 = 0$ and $V_{\mathrm{th}}^0 < 0$.

The Lloyd's bound is satisfied provided that $V_{\mathrm{th}}^0 \geq 0$.

The Lloyd's bound can be violated if $V_{\mathrm{th}}^0 < 0$.

We also find examples of black holes with divergent V_{th}^0, and interestingly enough, the $\dot{I}_{\mathrm{WDW}}$ diverges also, in the exactly the same way.

Further tests

STU gauged supergravities in $D=4$: [Cvetic et al. hep-th/9903214]

$$
\begin{aligned}
ds_4^2 &= -\prod_{i=1}^{4} H_i^{-\frac{1}{2}} \tilde{f}\, dt^2 + \prod_{i=1}^{4} H_i^{\frac{1}{2}} \left(\frac{dr^2}{\tilde{f}} + r^2 d\Omega_{2,k}^2 \right) , \\
A^i &= \frac{\sqrt{q_i(\mu + k q_i)}}{(r+q_i)} dt \,, \qquad X_i = H_i^{-1} \prod_{j=1}^{4} H_j^{\frac{1}{2}} \,, \\
\tilde{f} &= 1 - \frac{\mu}{r} + g^2 r^2 \prod_{i=1}^{4} H_i \,, \qquad H_i = 1 + \frac{q_i}{r} \,.
\end{aligned}
$$

Five all nonnegative integration constants $(\mu, q_1, q_2, q_3, q_4)$:

$$
\begin{aligned}
M &= \frac{1}{2}\mu + \frac{1}{4} k \sum_{j=1}^{4} q_j \,, \quad Q_i = \frac{1}{4}\sqrt{q_i(\mu + k q_i)} \,, \quad i = 1,2,3,4. \\
T &= \frac{\tilde{f}'}{4\pi} \prod_{j=1}^{4} H_j^{-\frac{1}{2}} \Big|_{r=r_0} \,, \qquad S = \pi \sqrt{\prod_{j=1}^{4} (r_0 + q_j)} \,, \\
\Phi_i &= \frac{\sqrt{q_i(\mu + k q_i)}}{r_0 + q_i} \,, \qquad P_{\text{th}} = \frac{3g^2}{8\pi} \,.
\end{aligned}
$$

STU black holes: two horizons

The thermodynamical volume is

$$V_{\text{th}} = V_{\text{th}}(r_0)\,, \qquad V_{\text{th}}(r) = \frac{1}{3}\pi r^3 \Big(\prod_{i=1}^{4} H_i\Big) \sum_{j=1}^{4} H_i^{-1}\,.$$

It is independent of g^2. Can check: when $q_1 q_2 q_3 q_4 \neq 0$, there are two horizons and $\dot{I}_{\text{WDW}} = \Phi_i Q_i\Big|_{+}^{-}$.

$$\dot{I}_{\text{WDW}} = 2P_{\text{th}}\,\Delta V_{\text{th}} + k(r_+ - r_-)\,, \quad \Delta V_{\text{th}} = V_{\text{th}}\Big|_{-}^{+}\,.$$

$$\begin{aligned}
&\dot{I}_{\text{WDW}} - 2M \\
&= -\frac{1}{4}\mu \sum_i \Big(\frac{r_-}{r_- + q_i} + \frac{q_i}{r_+ + q_i}\Big) \\
&-\frac{1}{4}k \sum_i \Big(\frac{q_i(2r_- + q_i)}{r_- + q_i} + \frac{q_i^2}{r_+ + q_i}\Big)\,.
\end{aligned}$$

$$Z = \frac{1}{T}\frac{\partial \dot{I}_{\text{WDW}}}{\partial S}\Big|_{Q_i, P_{\text{th}}} - \frac{4}{3} \geq 0\,.$$

STU black holes: one horizon

When $q_4 = 0$, there is only one horizon, and

$$V_{\text{th}}^0 = V_{\text{th}}(r=0) = \frac{1}{3}\pi q_1 q_2 q_3 > 0\,.$$

We find

$$\dot{I}_{\text{WDW}} = 2M - (\Phi_1 Q_1 + \Phi_2 Q_2 + \Phi_3 Q_3) - \frac{1}{4}k(q_1+q_2+q_3) - 2P_{\text{th}}V_{\text{th}}^0\,,$$

which implies that

$$\dot{I}_{\text{WDW}} = 2P_{\text{th}}\Delta V_{\text{th}} + kr_+\,, \qquad \Delta V_{\text{th}} = V_{\text{th}}(r_+) - V_{\text{th}}^0\,.$$

Can verify

$$\frac{1}{T}\frac{\partial \dot{I}_{\text{WDW}}}{\partial S}\Big|_{Q_i,P_{\text{th}}} > 1\,, \qquad k = 0, 1.$$

$$\frac{1}{T}\frac{\partial(2P_{\text{th}}\Delta V_{\text{th}})}{\partial S}\Big|_{Q_i,P_{\text{th}}} > 0\,, \qquad k = 1.$$

Negative volume and violation of Lloyd's bound

Einstein-scalar theory

$$\mathcal{L} = \sqrt{-g}\Big(R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\Big)\,,$$
$$V = -2g^2\Big((\cosh\phi + 2) - 2\beta^2(2\phi + \phi\cosh\phi - 3\sinh\phi)\Big).$$

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r(r+q)d\Omega_{2,k}^2\,, \qquad e^{\phi} = 1 + \frac{q}{r}\,,$$
$$f = g^2 r^2 + k - \frac{1}{2}g^2\beta^2 q^2 + g^2(1-\beta^2)qr + g^2\beta^2 r^2\Big(1+\frac{q}{r}\Big)\log\Big(1+\frac{q}{r}\Big)\,.$$

One integration constant q, paramterizing the mass of the black hole

$$M = \frac{1}{12}g^2\beta^2 q^3\,.$$

Black hole condition

$$\frac{1}{2}g^2\beta^2 q^2 - k > 0\,,$$

$$
\begin{aligned}
T &= \frac{f'(r_+)}{4\pi}\,, \qquad S = \pi r_+(r_+ + q)\,, \qquad P_{\text{th}} = \frac{3g^2}{8\pi}\,, \\
V_{\text{th}} &= \frac{2}{3}\pi r_+^3\Big(1+\frac{q}{r_+}\Big)\Big(2+\frac{q}{r_+}\Big)\left(1+\beta^2\log\Big(1+\frac{q}{r_+}\Big)\right) \\
&\quad -\frac{1}{9}\pi\beta^2 q\left(q^2+12qr_+ + 12r_+^2\right)\,.
\end{aligned}
$$

The volume of black hole singularity is negative!

$$
V_{\text{th}}^0 = -\frac{1}{9}\pi\beta^2 q^3 < 0\,.
$$

Indeed we can have also studied by [Guo and Fan 1811.01473]

$$
\dot{I}_{\text{WDW}} = 3M - \frac{1}{2}kq = 2M - \frac{1}{2}kq - 2P_{\text{th}}V_{\text{th}}^0 < 2M\,.
$$

But our proposals remain true:

$$
\dot{I}_{\text{WDW}} - 2P_{\text{th}}\,\Delta V_{\text{th}} = kr_+\,.
$$

$$
\frac{1}{T}\frac{\partial \dot{I}_{\text{WDW}}}{\partial S}\Big|_{P_{\text{th}}} = 3 - \frac{2k}{\beta^2 g^2 q^2} \qquad \begin{cases} \ge 2, & k=1\,, \\ = 3, & k=0\,. \end{cases}
$$

Summary of our investigation

We have examined a diverse selections of AdS black holes in Einstein gravities satisfying the null energy conditions. (Nine different classes of black holes.) We find that

- Lloyd's bound can be violated and the violation is related to the negative volume of the black hole singularity.
- The inequality $\frac{1}{T}\frac{\partial(2P_{\text{th}}\Delta V_{\text{th}})}{\partial S}\Big|_{Q_i,P_{\text{th}}} > 0$ for $k=1$ can be violated. The underlying reason is not clear.
- The following relations

  $$\frac{1}{T}\frac{\partial \dot{I}_{\text{WDW}}}{\partial S}\Big|_{Q,P_{\text{th}}} > C\,,$$

  and

  $$\begin{cases} \dot{I}_{\text{WDW}} > 2P_{\text{th}}\,\Delta V_{\text{th}}, & k=1, \\ \dot{I}_{\text{WDW}} = 2P_{\text{th}}\,\Delta V_{\text{th}}, & k=0, \\ \dot{I}_{\text{WDW}} < 2P_{\text{th}}\,\Delta V_{\text{th}}, & k=-1. \end{cases}$$

  always hold. The smallest C we find is $C=(D-3)/(D-2)$.

Conclusions

- We studied CA conjecture $\mathcal{C}_A = I_{\text{WDW}}$ and introduced a new CV conjecture $\dot{\mathcal{C}}_V = 2P_{\text{th}}\Delta V_{\text{th}}$.

- Two horizons: $\Delta V_{\text{th}} = V_{\text{th}}\Big|_{-}^{+}$; one horizon: $\Delta V_{\text{th}} = V_{\text{th}}^{+} - V_{\text{th}}^{0}$.

- We find following relations

$$\begin{cases} \dot{\mathcal{C}}_A > \dot{\mathcal{C}}_V, & k = 1, \\ \dot{\mathcal{C}}_A = \dot{\mathcal{C}}_V, & k = 0, \\ \dot{\mathcal{C}}_A < \dot{\mathcal{C}}_V, & k = -1. \end{cases}$$

  and

$$\frac{1}{T}\frac{\partial \dot{\mathcal{C}}_A}{\partial S} > C\,, \quad k = 0, 1.$$

  The differential lower bound is holographic conclusion that is worth deriving in quantum information theory directly.

- Even if both the CA and new CV conjectures are incorrect, our inequality relations may have revealed something deep in black hole physics.

- Our volume of spacetime singularity also deserves better understanding.