Feynman integrals from bottom up

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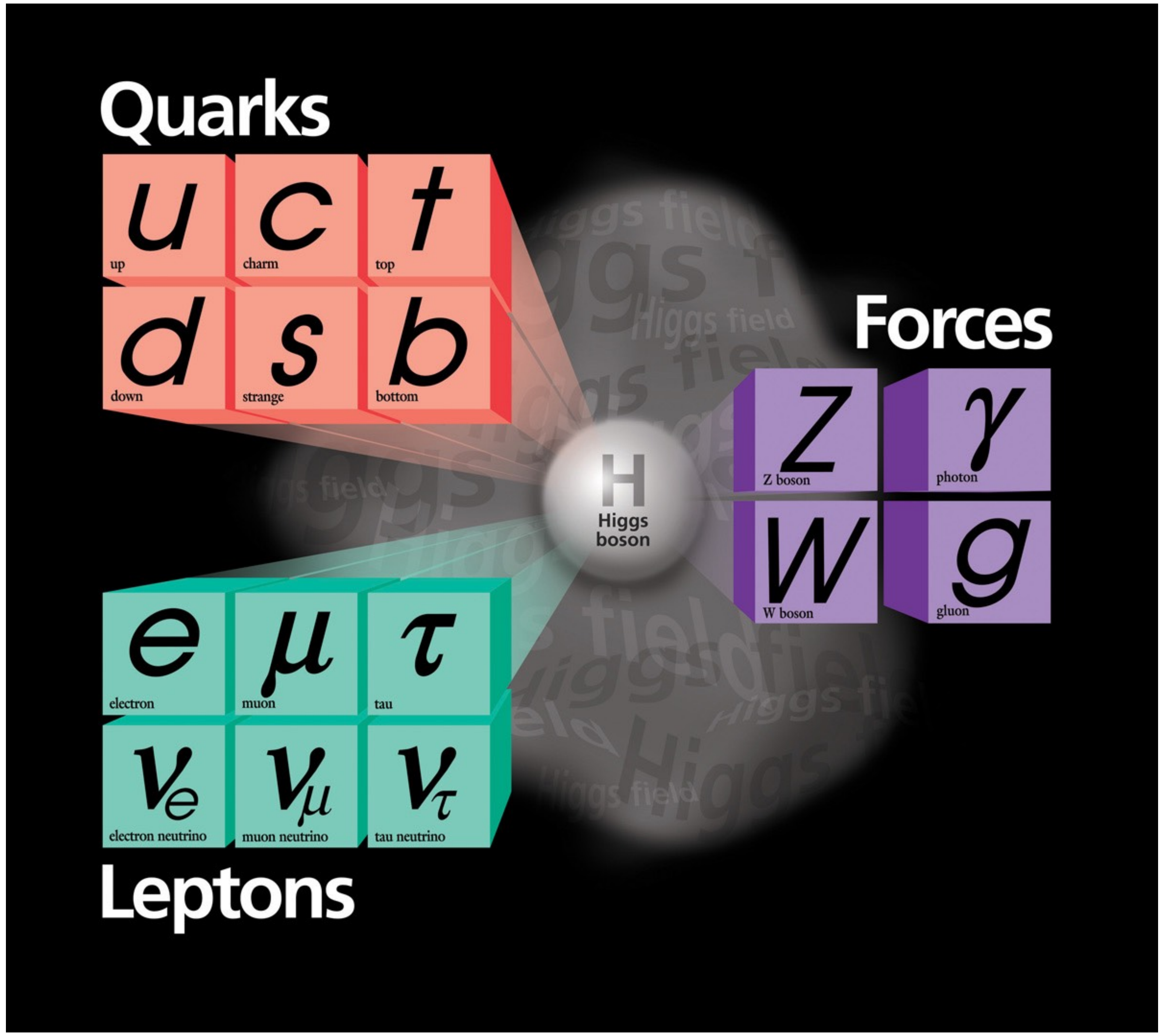
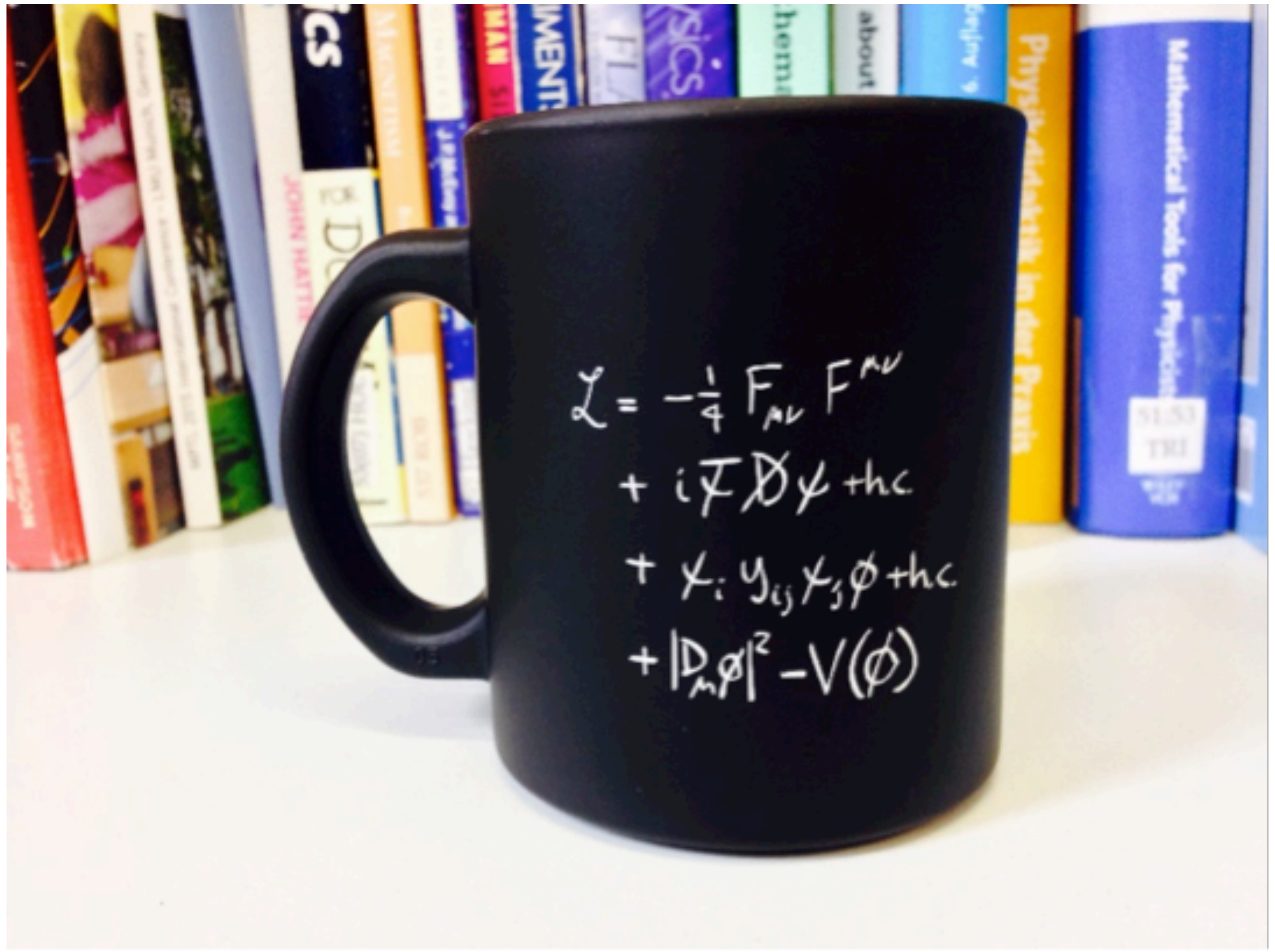
The Standard Model (SM) of particle physics

Our current understanding of fundamental constitutes of matter and their interactions



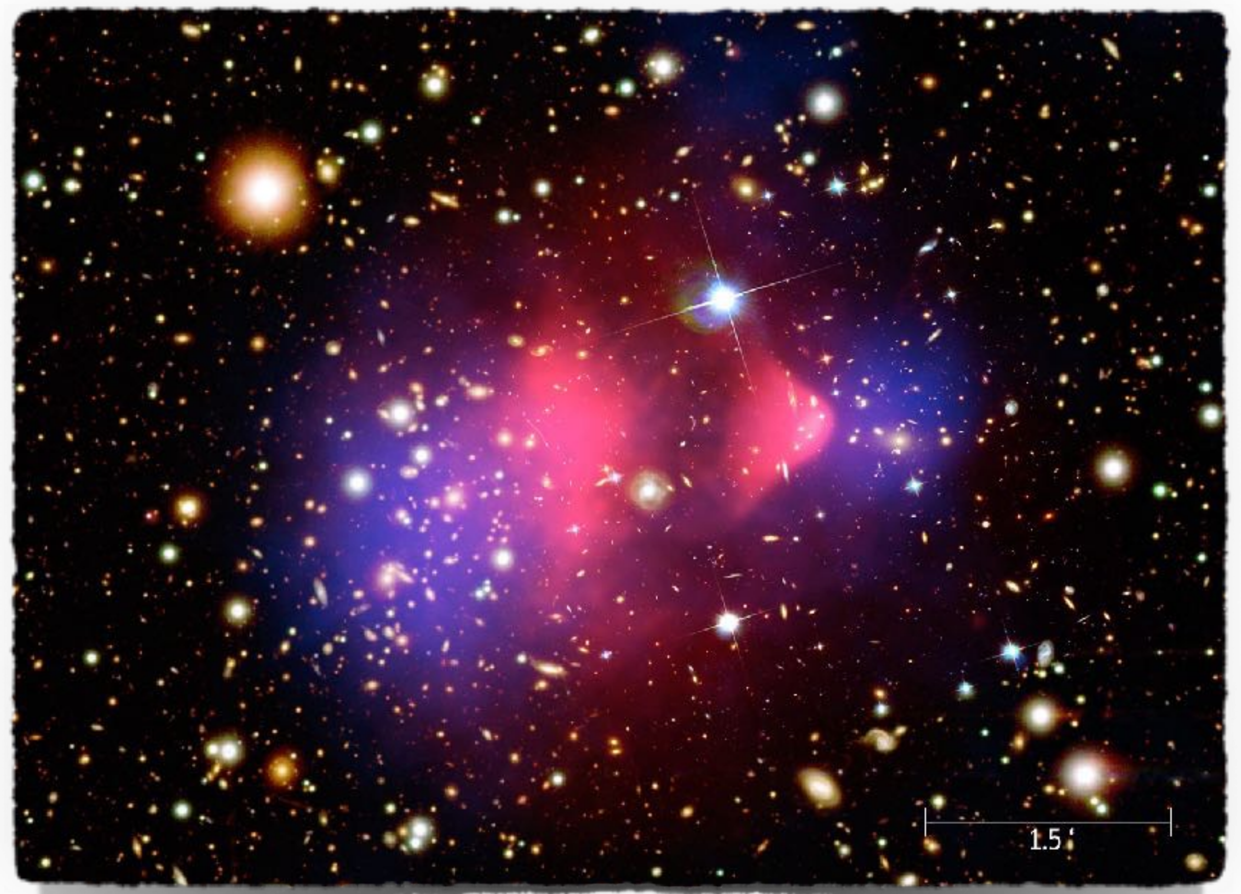


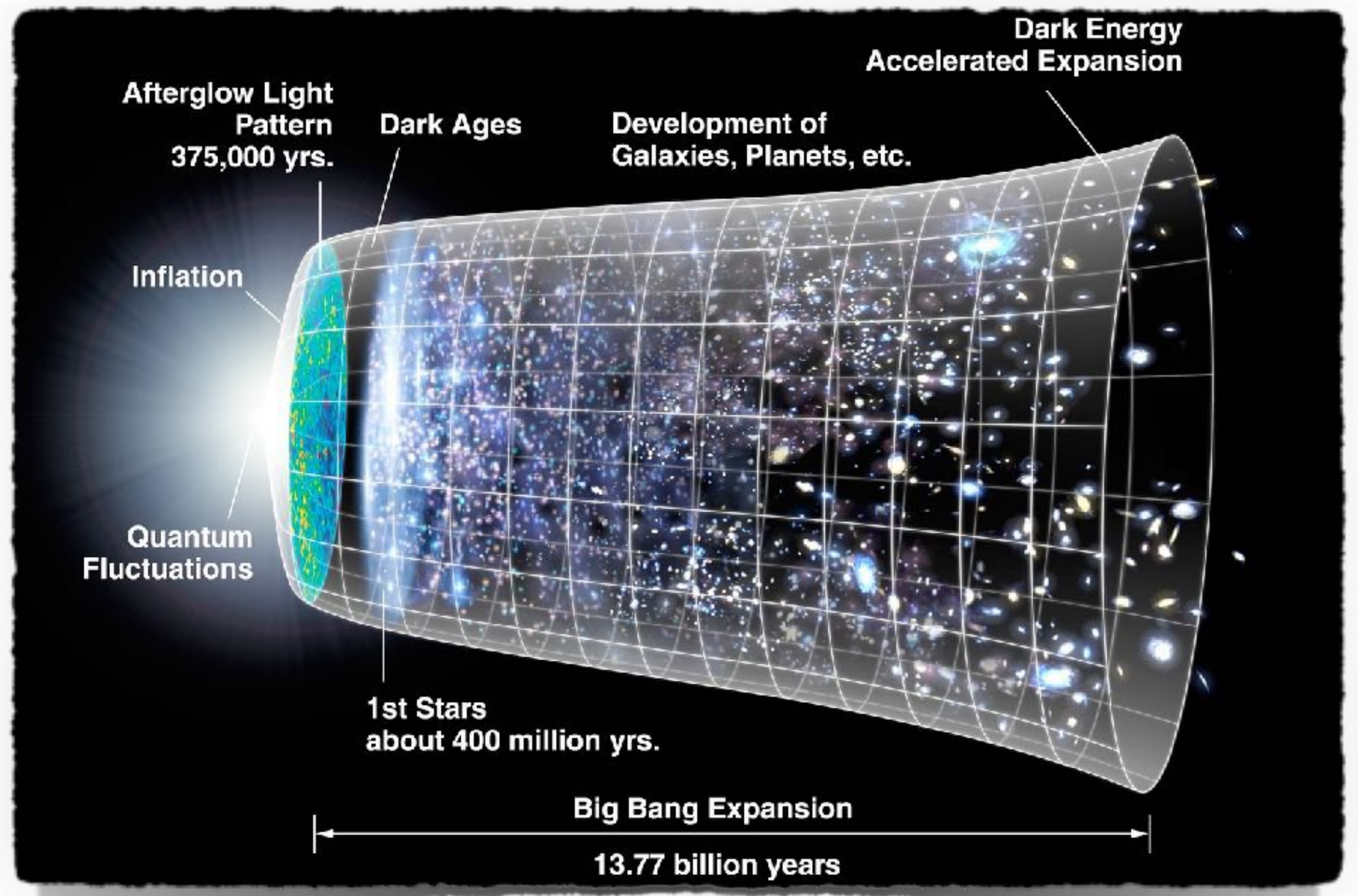
Built upon quantum gauge field theories

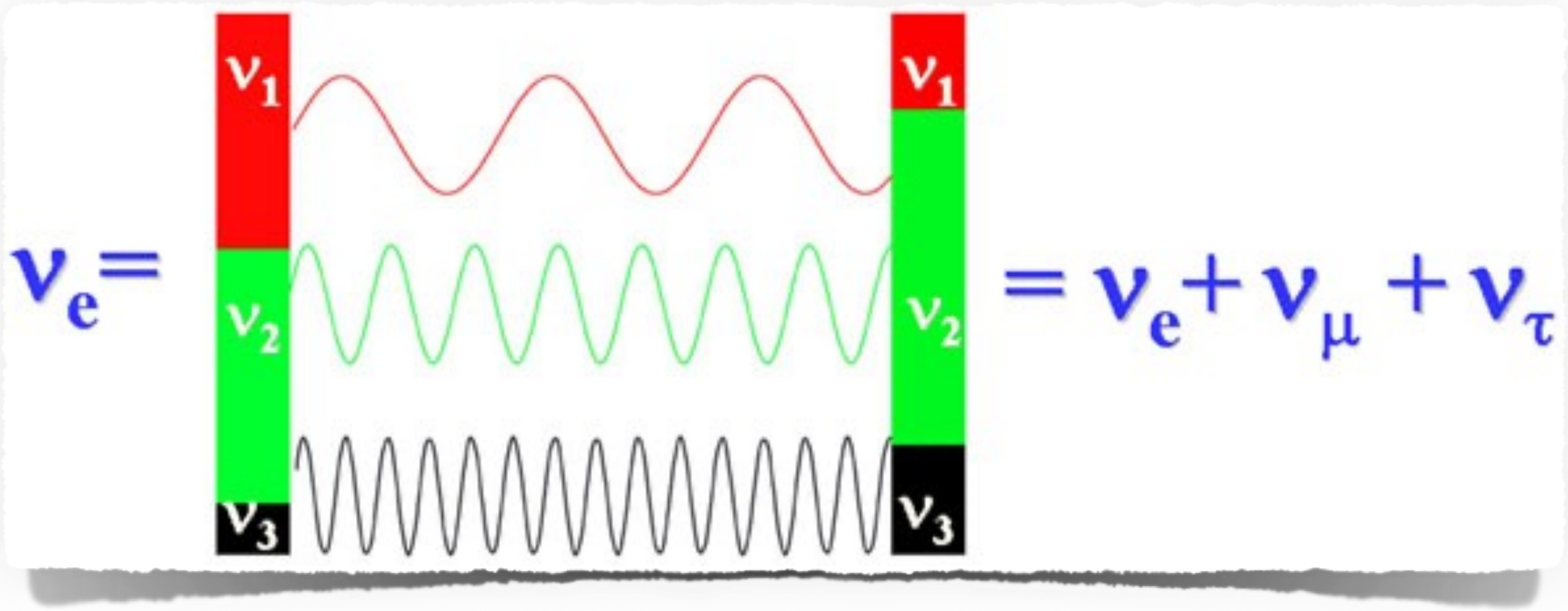
(in particular, Yang-Mills theories)

What's beyond the SM?

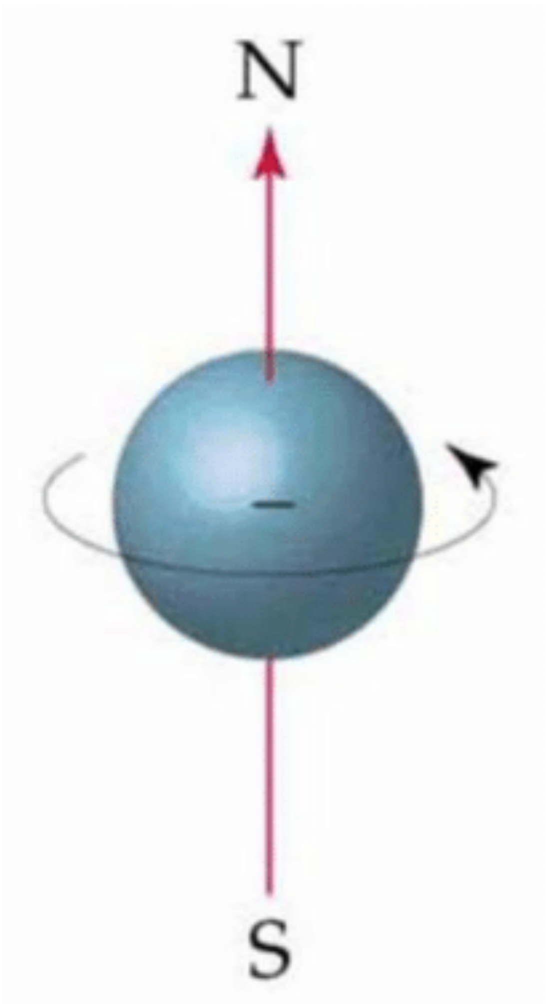
We know that there has to be something new at higher energies beyond the SM





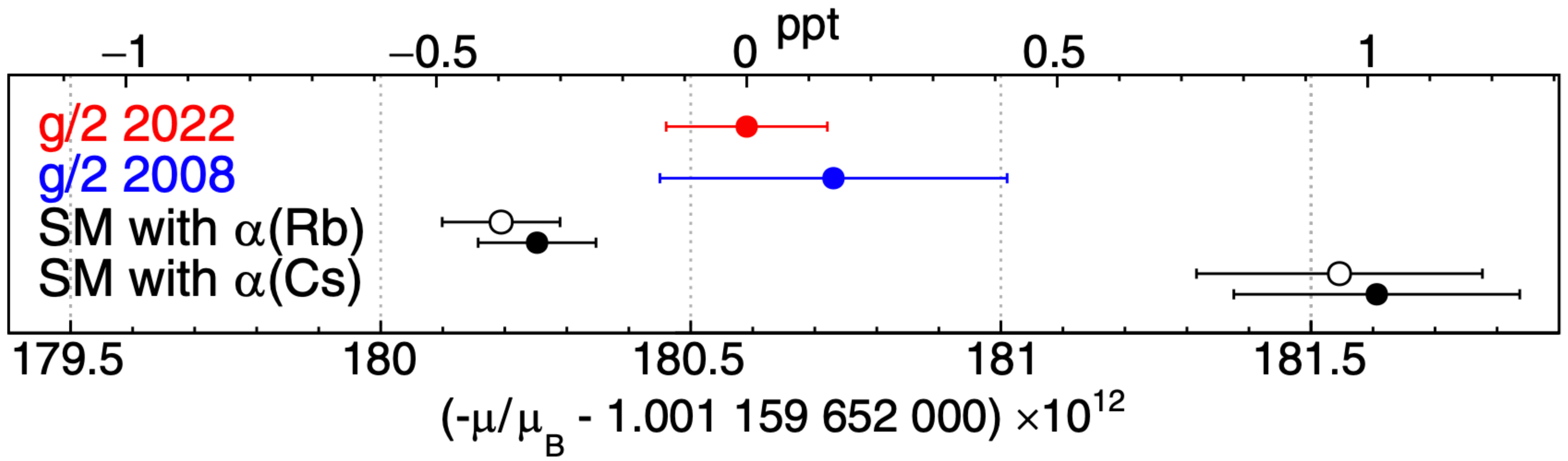


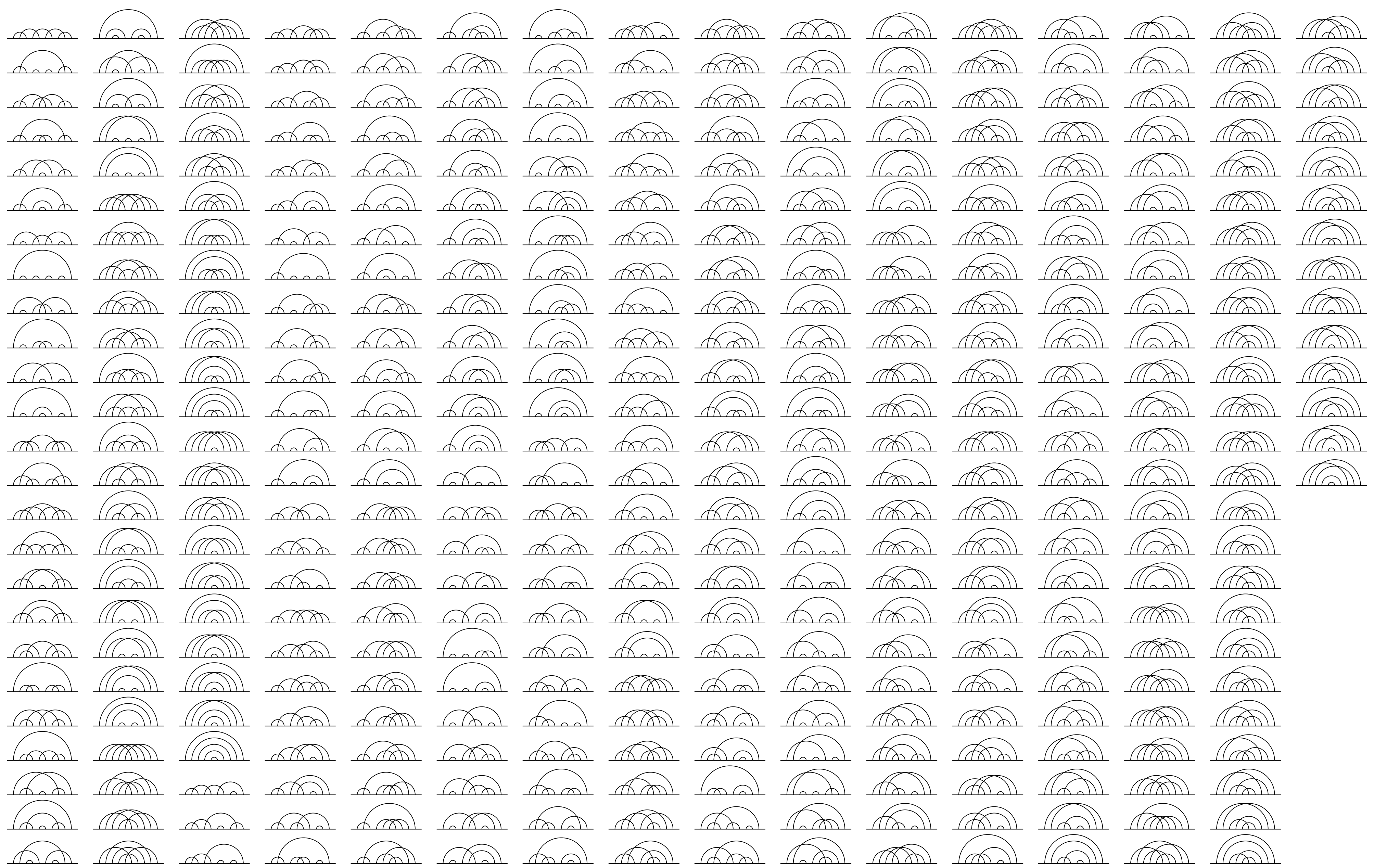
Precision tests of the SM: electron g-2



$$a_e^{\mathrm{EXP}} = 0.001\,159\,652\,180\,59(13)$$ Fan et al. (2022)

$$a_e^{\mathrm{SM}} = 0.001\,159\,652\,181\,606(11)(12)(229)$$ Aoyama et al. (2019) and a lot of efforts!





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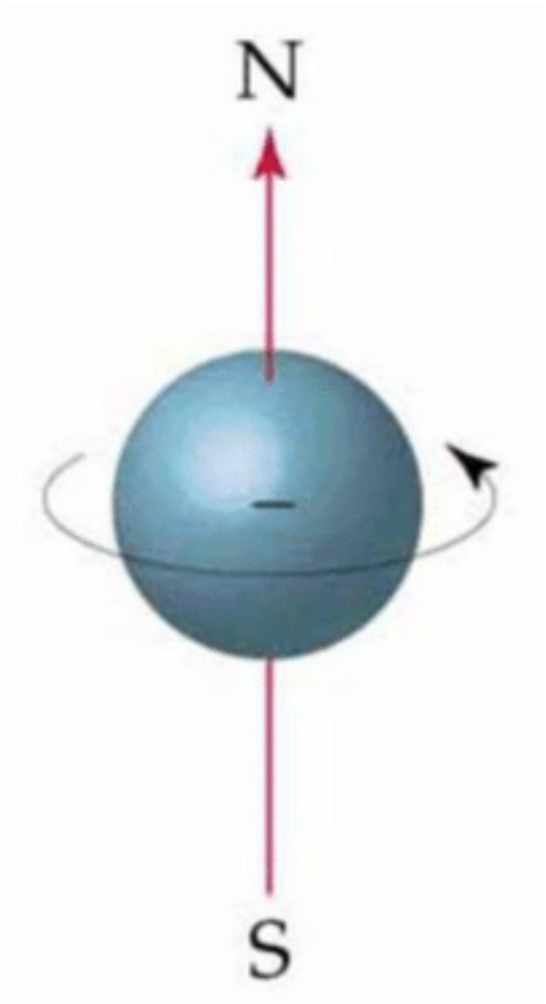

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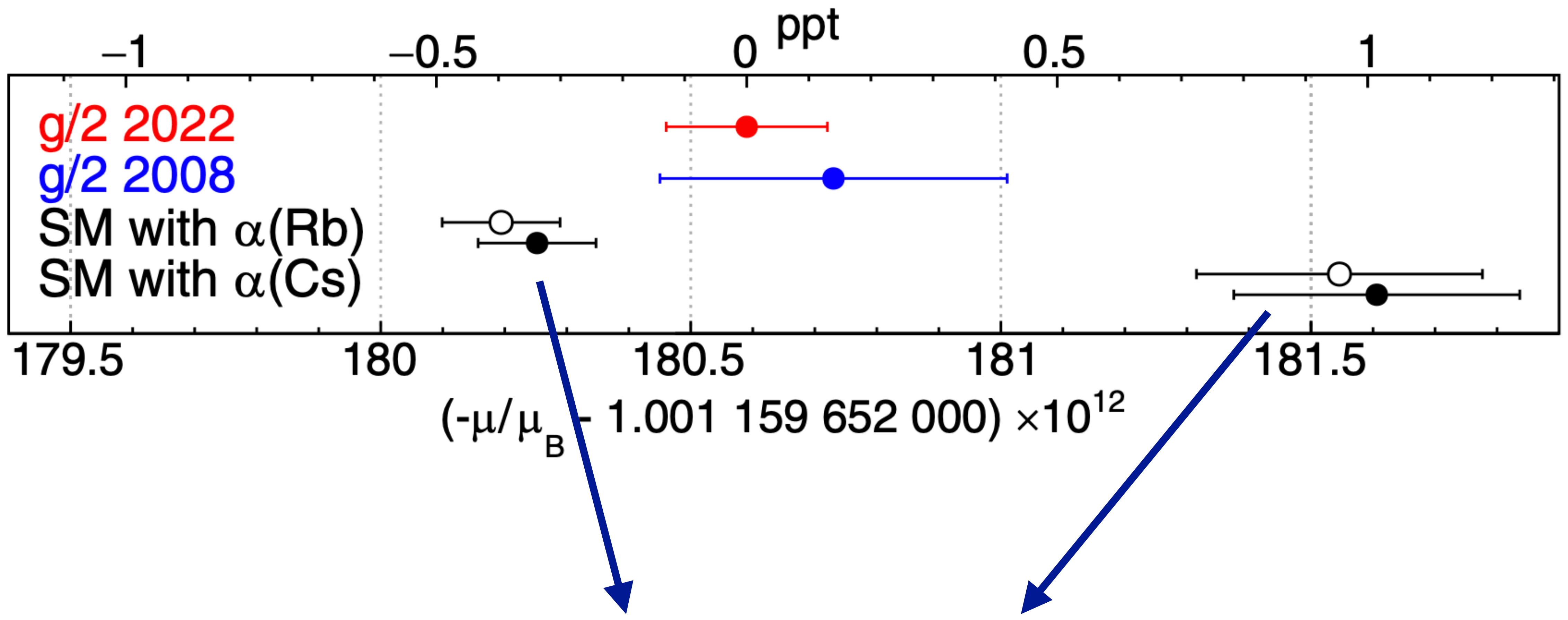
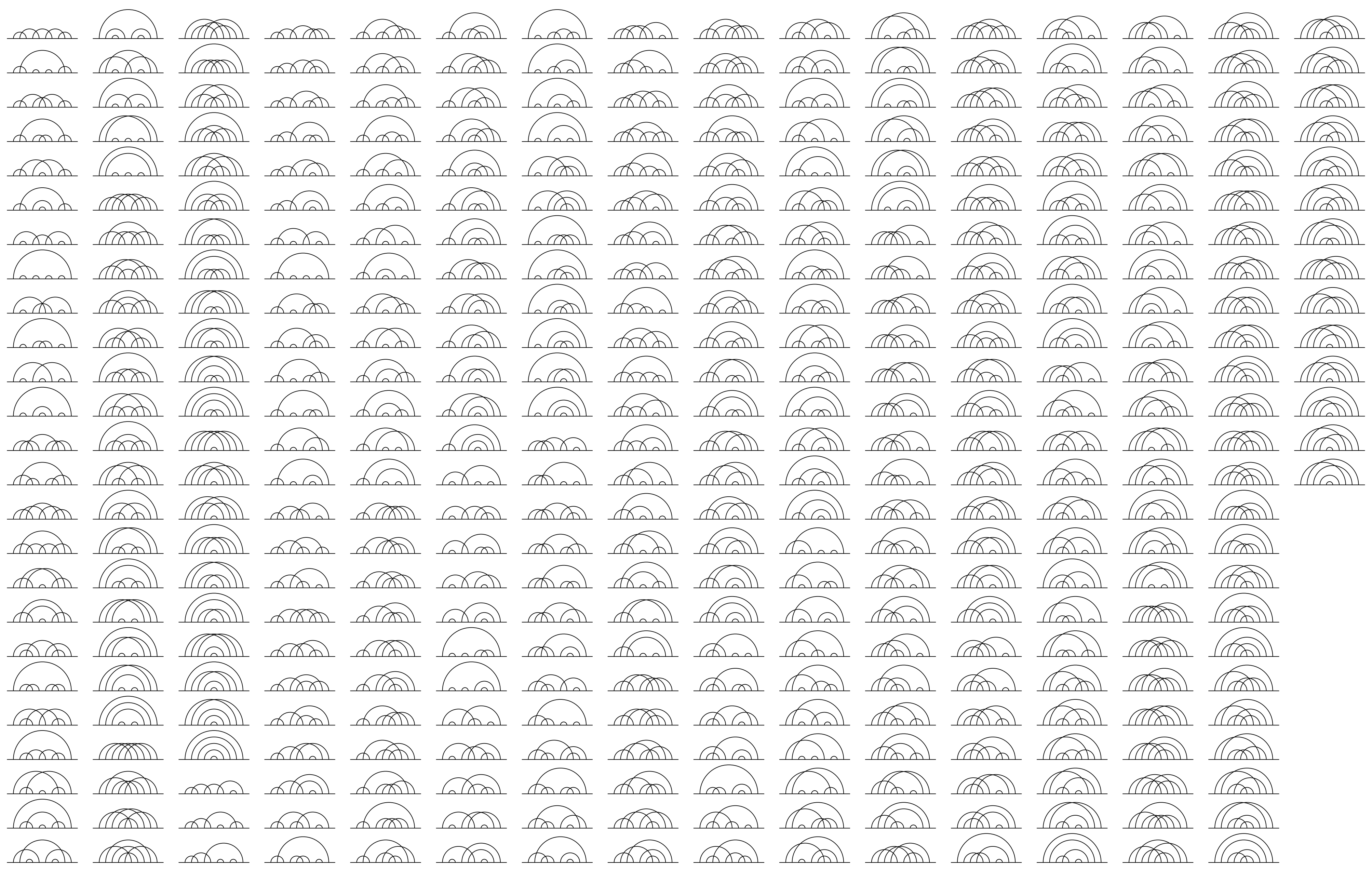
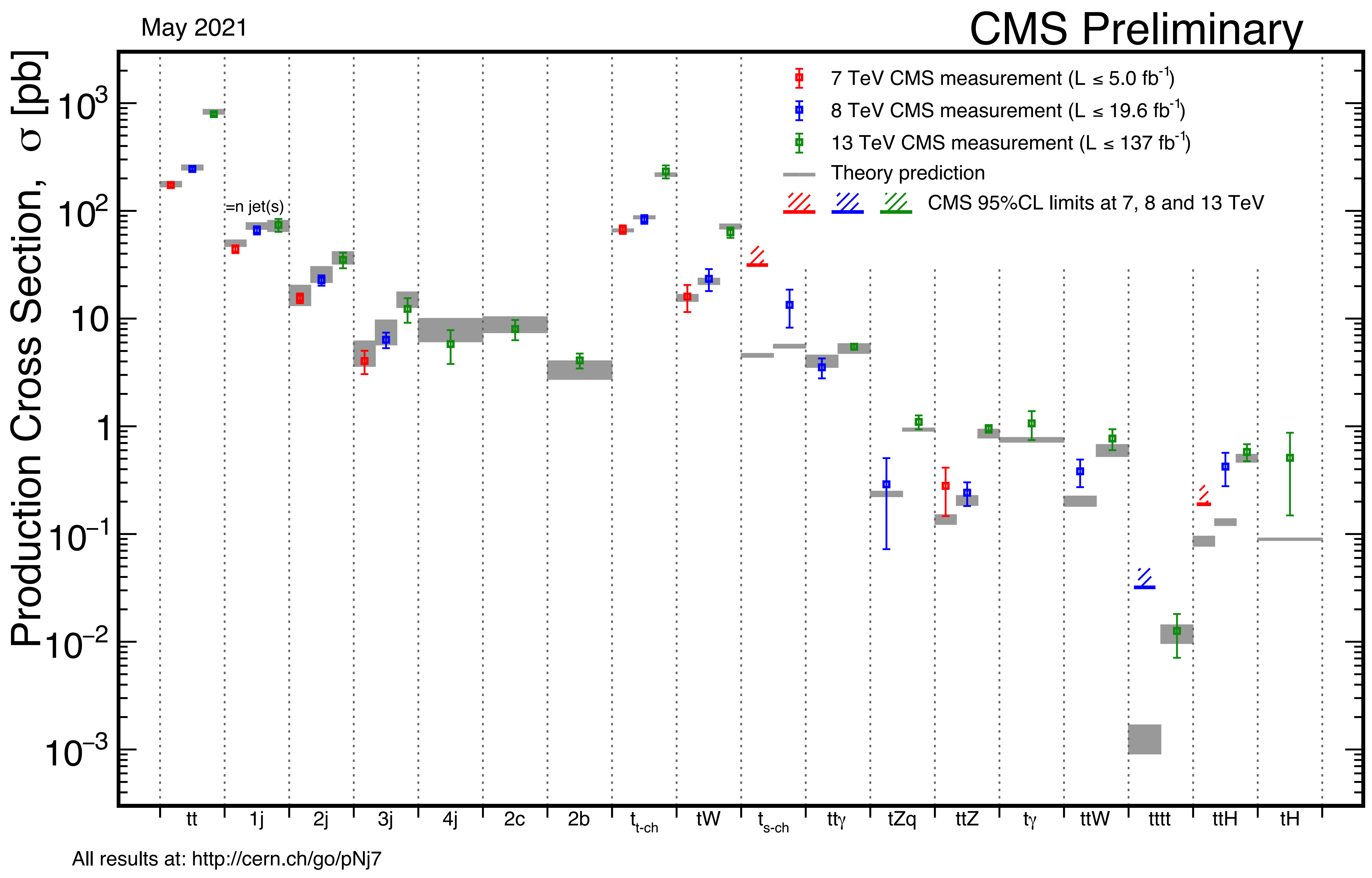
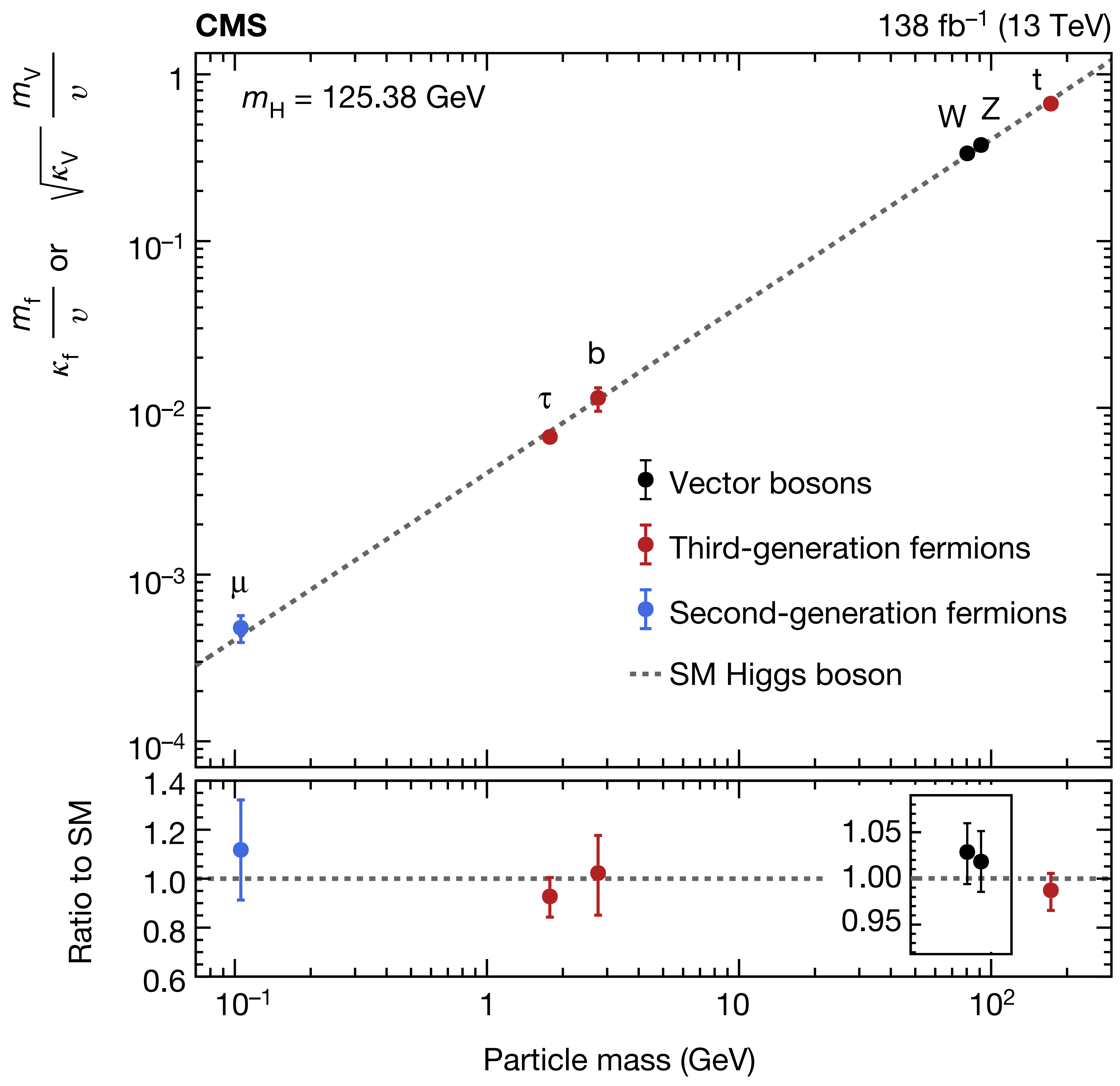




Calls for better understanding measurements of the fine-structure-constant in atomic physics

Precision tests of the SM: Large Hadron Collider

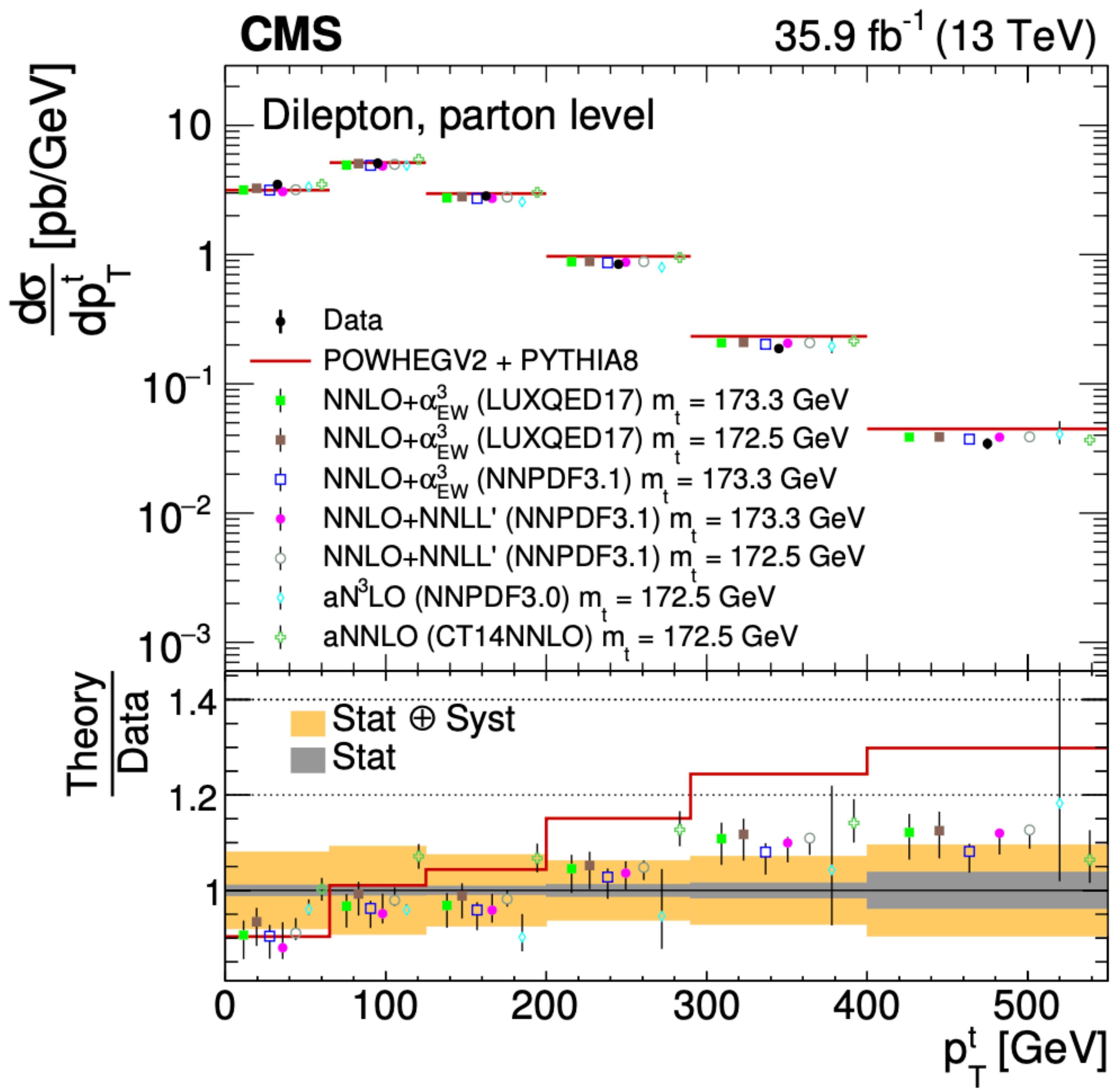


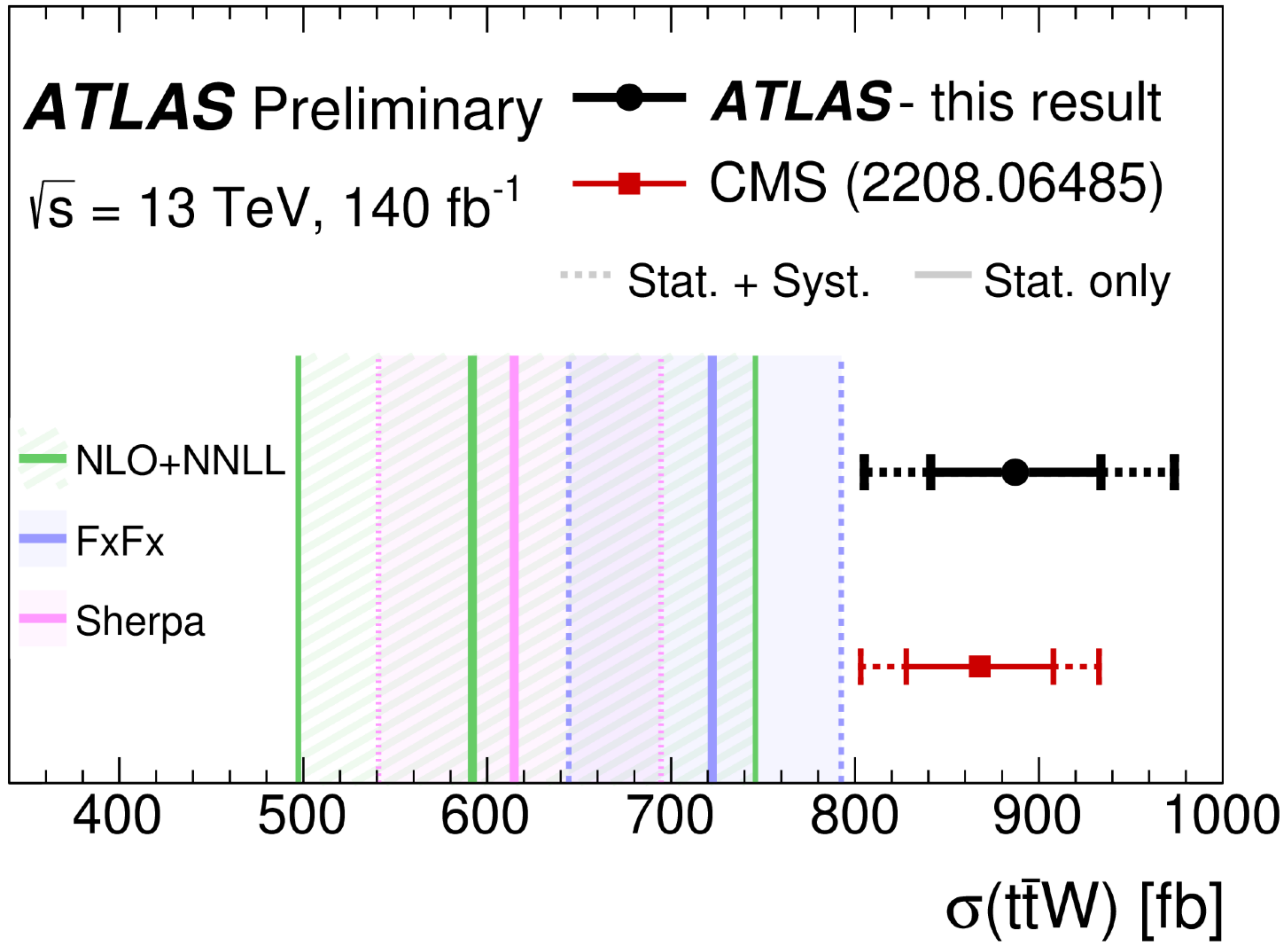


The LHC is testing the SM at unprecedented energies and precisions!

Backed up by developments in theoretical calculations during the past decades….

Precision tests need precision calculations





Precision tests need precision calculations

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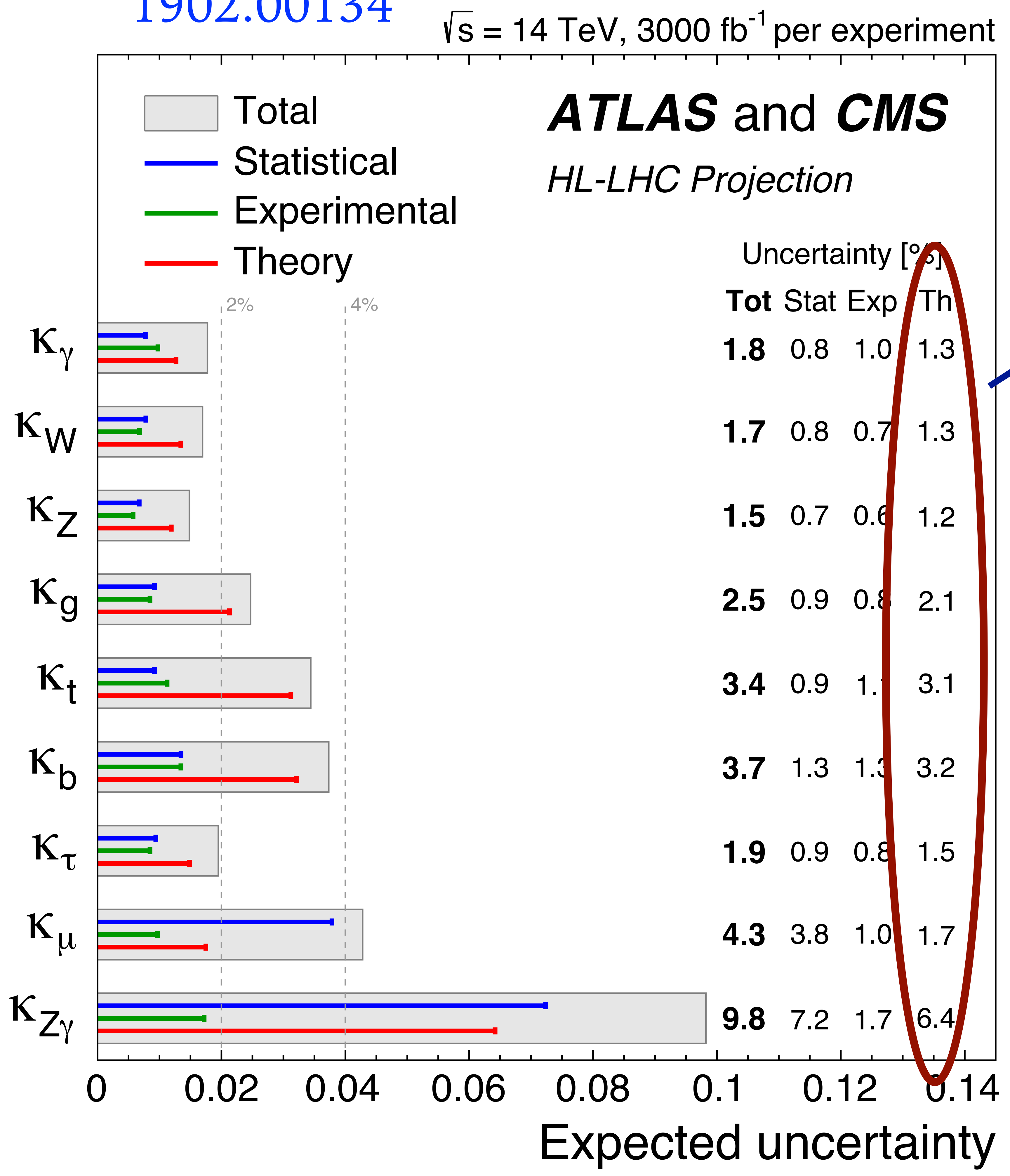
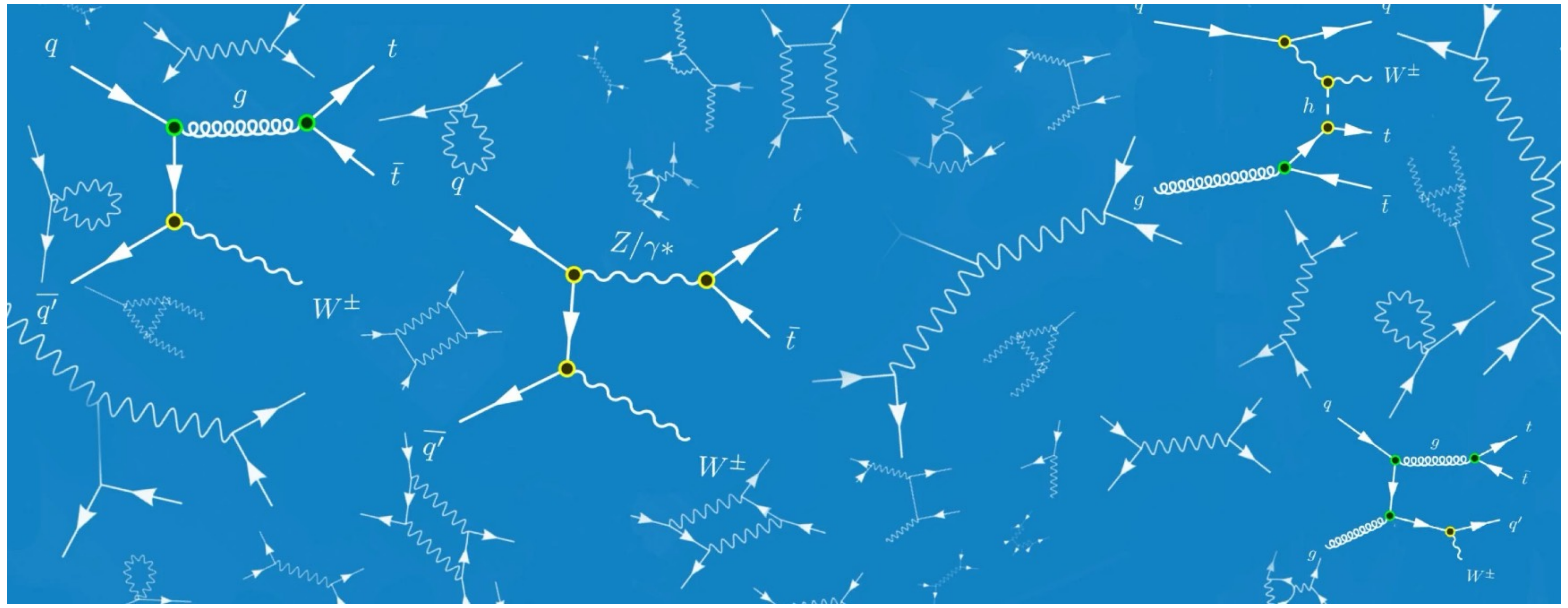


The upcoming experimental accuracies are demanding **much better** theoretical precision for various scattering processes

A lot of theoretical efforts going on

- ➤ Analytical methods
- ➤ Numerical methods
- ➤ Mathematical tools
- ➤ Phenomenological applications

Scattering amplitudes



Connecting theories and experiments

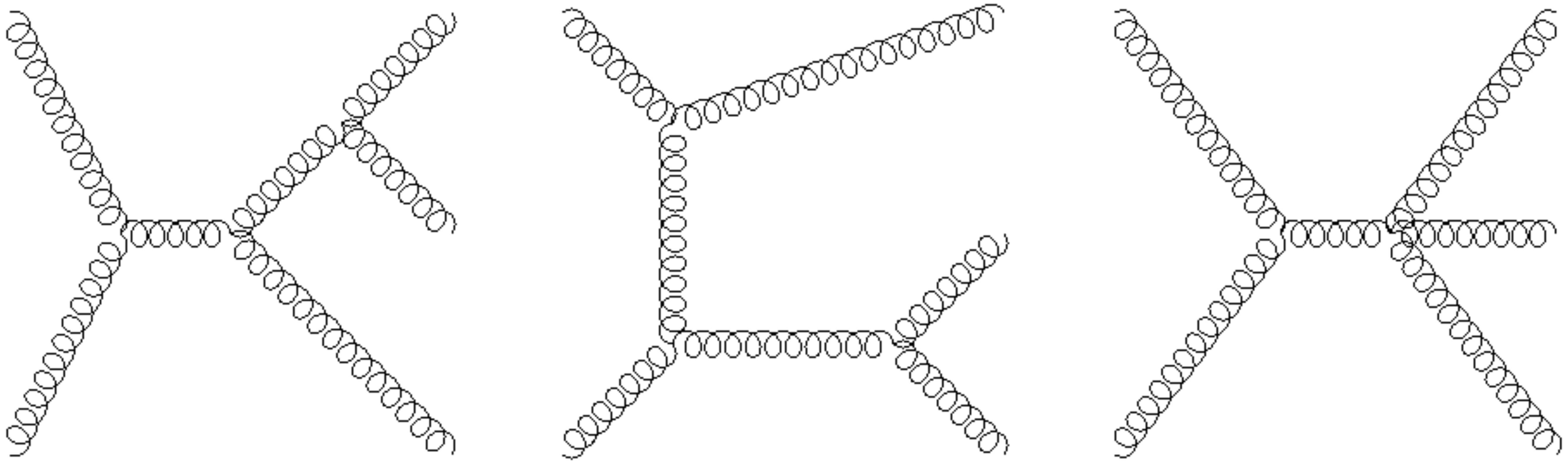
- ➤ Collider physics
- ➤ Dark matter direct/indirect searches
- ➤ Gravitational waves
- ➤ Cosmology

Revealing new structures of QFTs

Tree-level structures

Surprising insights from tree-level calculations:
complicated amplitudes can be made simple if

- We know the **correct language** to describe them
- We know how they come from **simple building blocks**

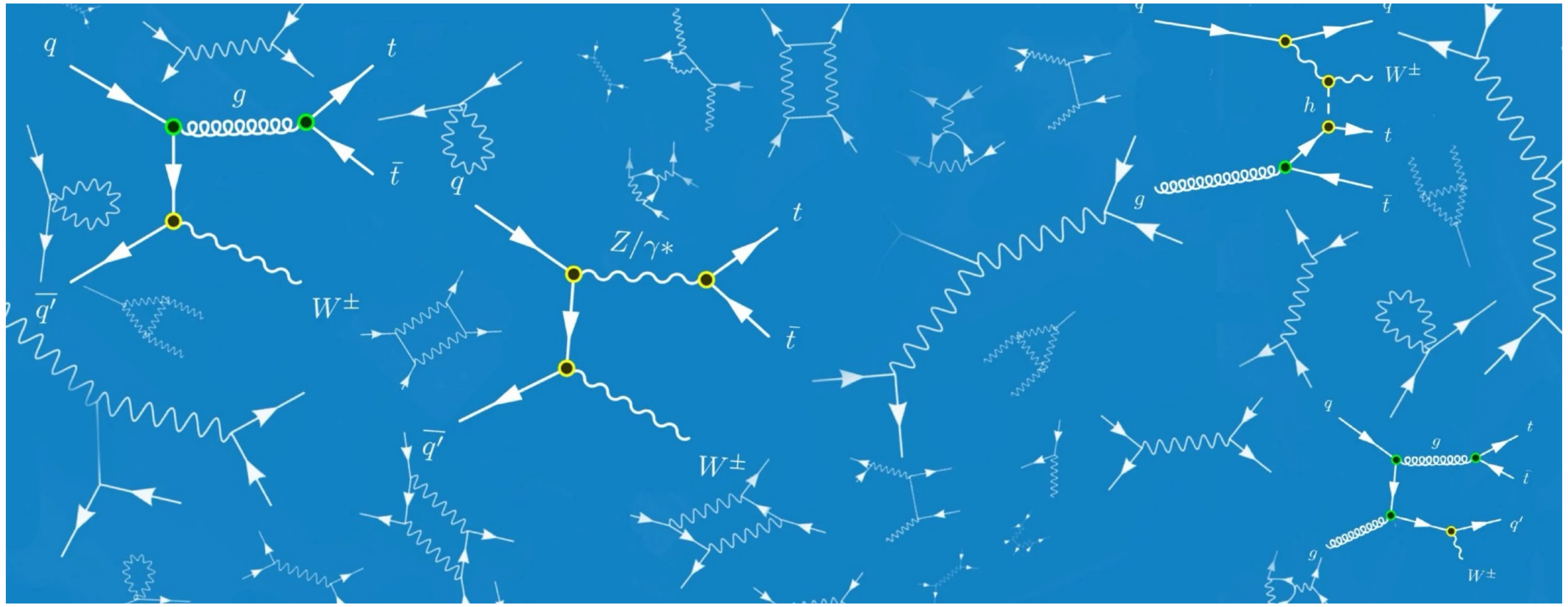


$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle (n-1)\, n\rangle\langle n\, 1\rangle}$$

Parke-Taylor (1986)
Xu-Zhang-Chang (1987)
BCFW (2005)

Many developments not covered here!

Loop-level amplitudes



Loop integrands

Loop integrals

Loop amplitudes

Modern analytic techniques for loop integrals

See, e.g., Weinzierl (2022)
and references therein

IBP reduction → Differential Equations → Canonical basis → Canonical DEs → Solutions

$$\mathrm{d}\vec{f}(\boldsymbol{x},\epsilon) = \epsilon\, \mathrm{d}\alpha_i(\boldsymbol{x})\, \boldsymbol{A}_i\, \vec{f}(\boldsymbol{x},\epsilon)$$

- Routine for polylogarithmic integrals
- Extending to elliptic integrals and more

Iterated integrals and symbol letters

The calculations based on canonical differential equations show that loop integrals can be represented in terms of Chen's iterated integrals

$$\int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \mathrm{d}\alpha_{i_n}(\boldsymbol{x}_n) \cdots \int_{\boldsymbol{x}_0}^{\boldsymbol{x}_3} \mathrm{d}\alpha_{i_2}(\boldsymbol{x}_2) \int_{\boldsymbol{x}_0}^{\boldsymbol{x}_2} \mathrm{d}\alpha_{i_1}(\boldsymbol{x}_1)$$

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Integration kernels = symbol letters

Iterated integrals and symbol letters

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Integration kernels = symbol letters

Encode lots of information about Feynman integrals

Iterated integrals and symbol letters

The calculations based on canonical differential equations show that loop integrals can be represented in terms of Chen's iterated integrals

$$\int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \mathrm{d}\alpha_{i_n}(\boldsymbol{x}_n)\cdots\int_{\boldsymbol{x}_0}^{\boldsymbol{x}_3} \mathrm{d}\alpha_{i_2}(\boldsymbol{x}_2)\int_{\boldsymbol{x}_0}^{\boldsymbol{x}_2} \mathrm{d}\alpha_{i_1}(\boldsymbol{x}_1)$$

Integration kernels = symbol letters

Encode lots of information about Feynman integrals

- ➤ The correct language?
- ➤ Simple building blocks?

Symbol letters

$$\int_{x_0}^{x} \mathrm{d}\alpha_{i_n}(x_n)\cdots\int_{x_0}^{x_3} \mathrm{d}\alpha_{i_2}(x_2)\int_{x_0}^{x_2} \mathrm{d}\alpha_{i_1}(x_1) \quad \longrightarrow \quad \alpha_{i_1}\otimes\alpha_{i_2}\otimes\cdots\otimes\alpha_{i_n}$$

Symbol map

Goncharov et al. (2010)
Duhr et al. (2011)

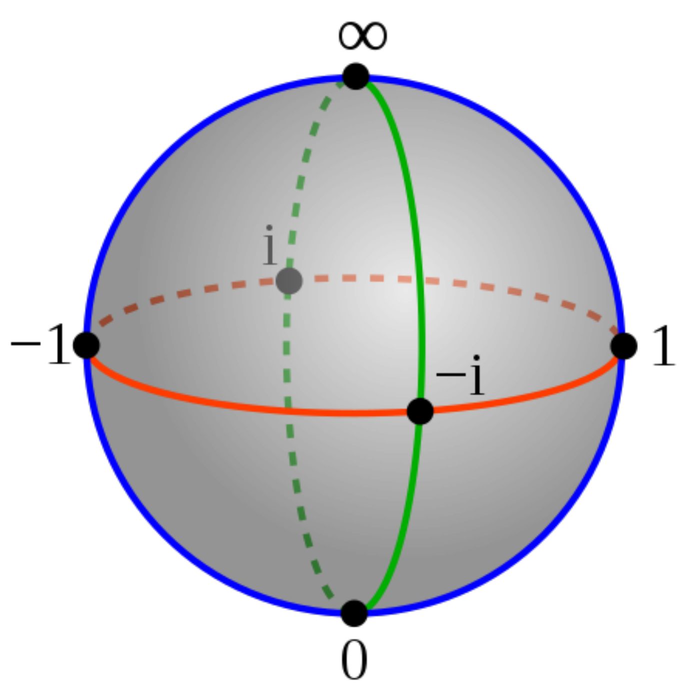
Analytic information: singularities determined by letters

Algebraic information, e.g., shuffle algebra:

$$(a\otimes b)\sqcup\!\sqcup\,(c\otimes d) = a\otimes b\otimes c\otimes d + a\otimes c\otimes b\otimes d + c\otimes a\otimes b\otimes d$$
$$+\, a\otimes c\otimes d\otimes b + c\otimes a\otimes d\otimes b + c\otimes d\otimes a\otimes b\,.$$

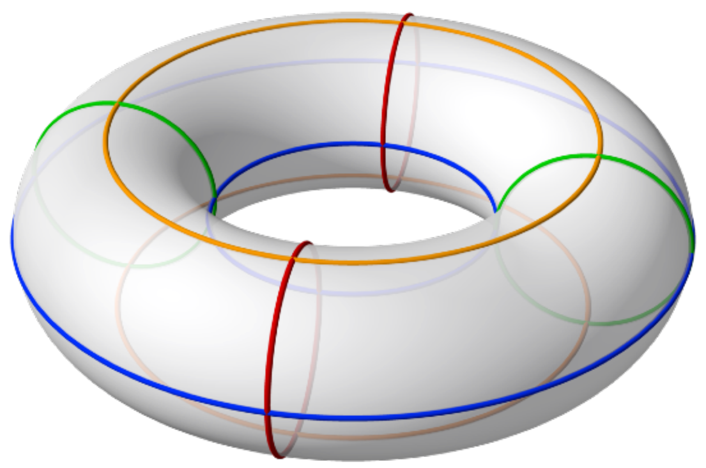
Symbol letters

Geometric information



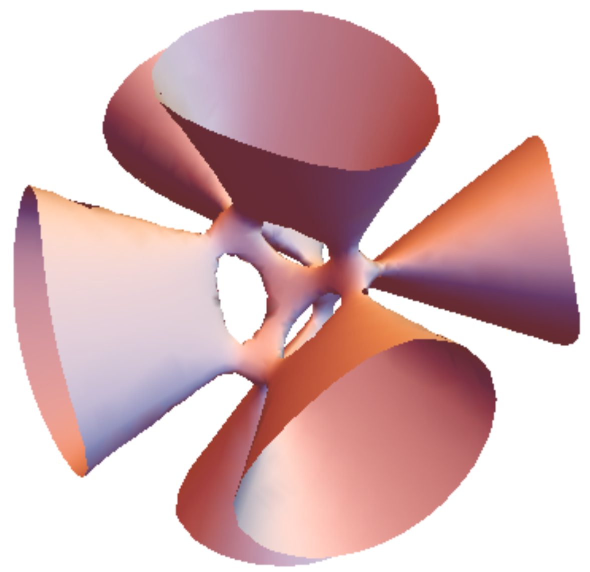
Polylogarithmic integrals

$$\alpha_i(\boldsymbol{x}) = \log W_i(\boldsymbol{x})$$



Elliptic integrals, modular forms…

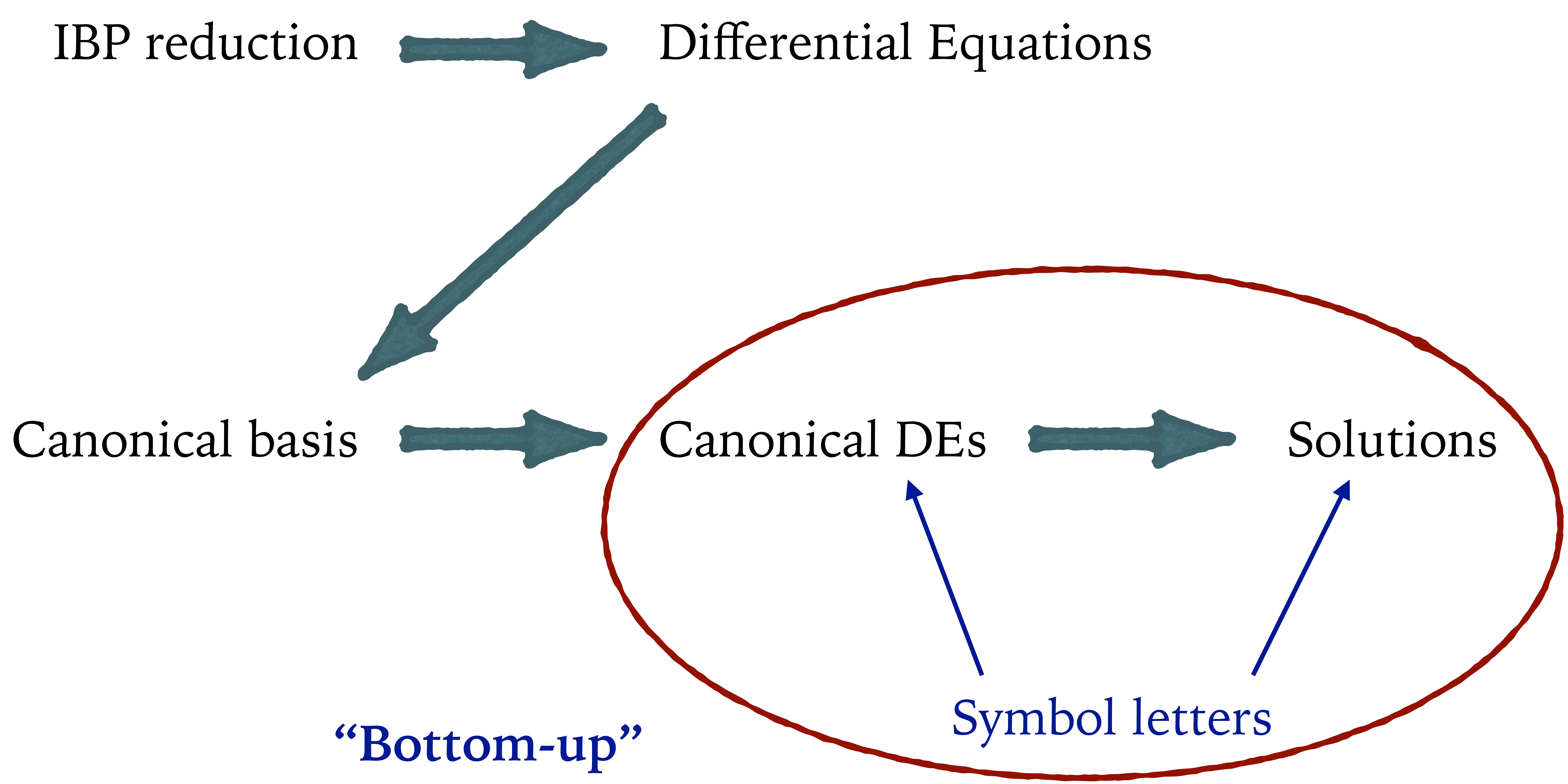
Still calling for better understanding



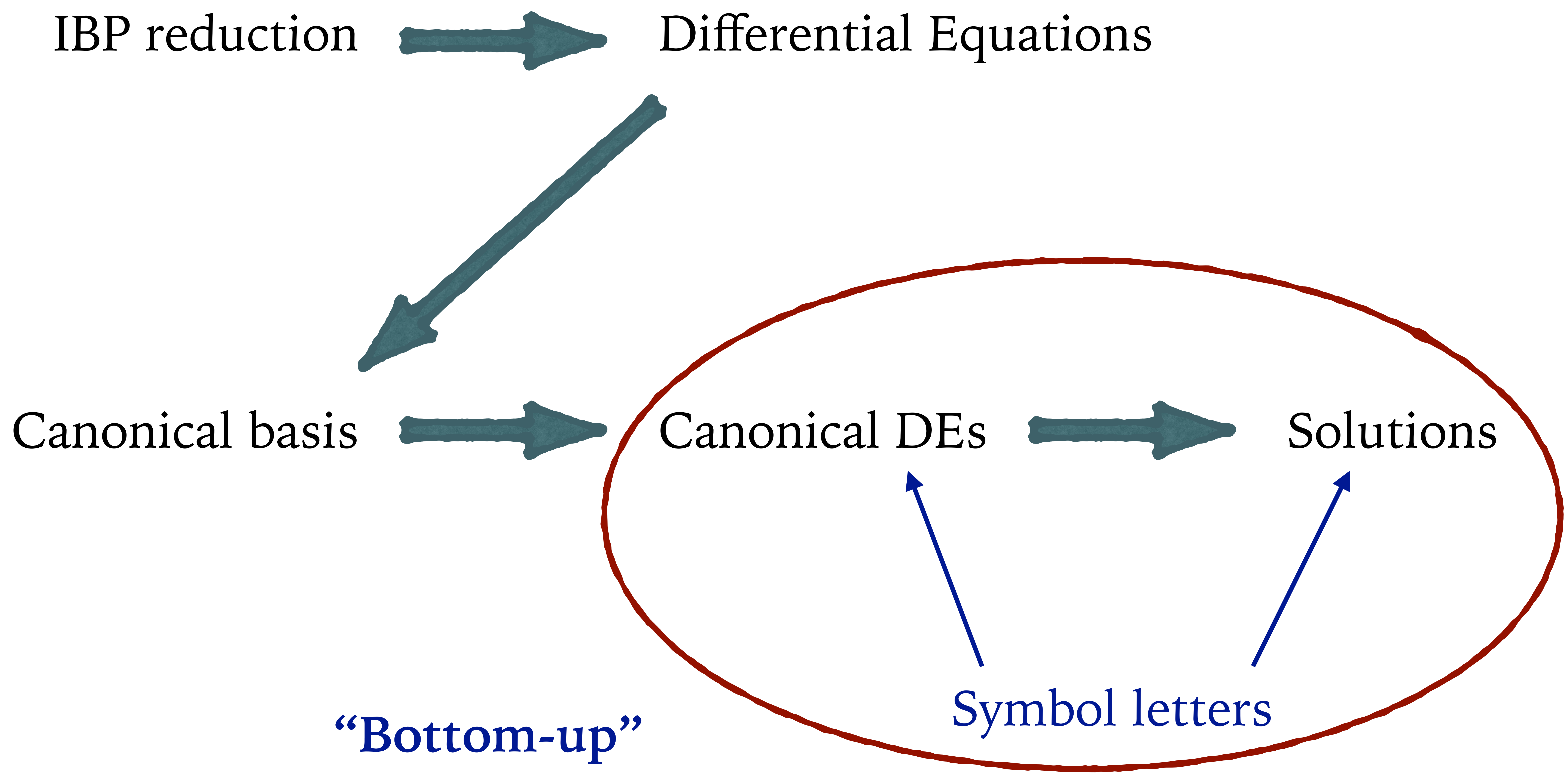
More complicated manifolds

Studies emerging!

From symbol letters to loop integrals



From symbol letters to loop integrals



Try to understand the symbol letters using Baikov representations + intersection theory

Baikov representations

Baikov (1996)

Change of variables from loop momenta to propagator denominators

$$\int \left[\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}}\right] \frac{1}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}} \longrightarrow \int_{\mathscr{C}} u(z) \frac{\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{z_1^{a_1} \cdots z_n^{a_n}}$$

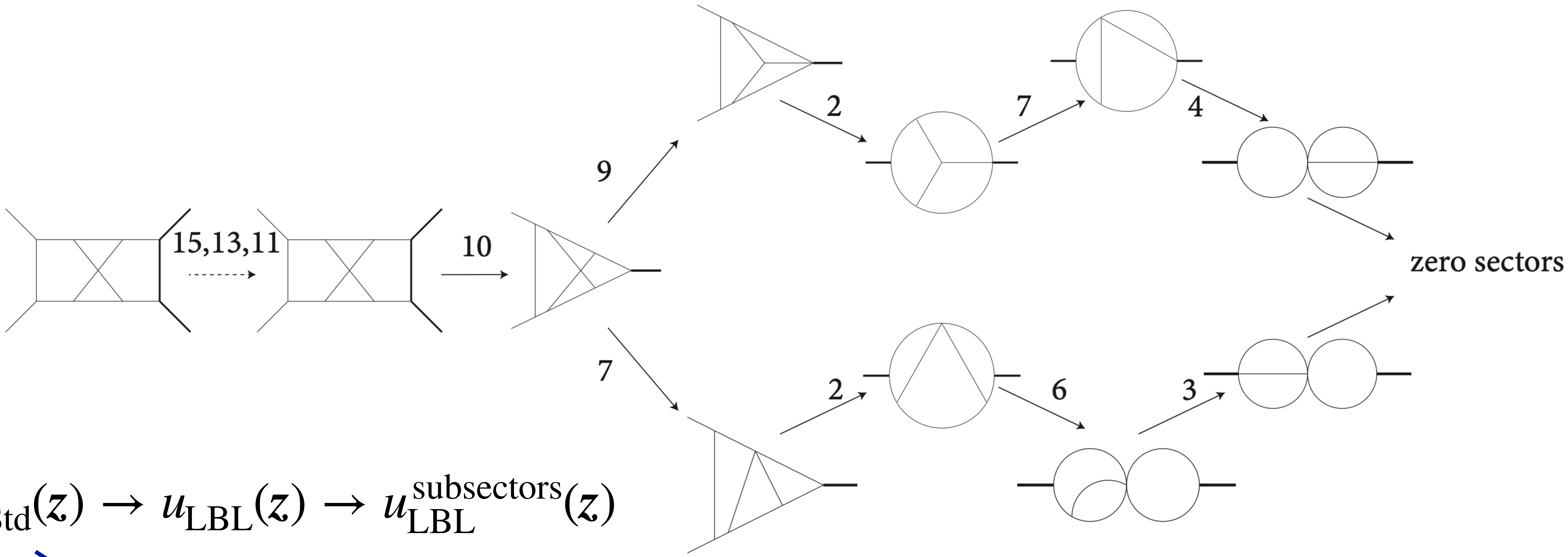
$$z_m = \sum_{i,j} A_m^{ij} q_i \cdot q_j + f_m$$

Contains all information about an integral family (including all sub-sectors)

$$u(z) = \left[P_1(z)\right]^{\gamma_1} \cdots \left[P_m(z)\right]^{\gamma_m}$$

Recursive structure of Baikov representations

Jiang, LLY (2023)



$u_{\text{Std}}(z) \to u_{\text{LBL}}(z) \to u_{\text{LBL}}^{\text{subsectors}}(z)$

All u-functions can be generated from a single one

Geometric formulation of IBP equivalence

Frellesvig et al. (2019)

The u-functions generate IBP relations among integrals

$$I = \int_{\mathscr{C}} u(z)\,\varphi(z)$$

→ n-form

$$0 = \int_{\mathscr{C}} d(u(z)\xi(z)) = \int_{\mathscr{C}} u(z)\,\nabla_\omega \xi(z)$$

↙ $(n-1)$-form

$$\nabla_\omega \equiv d + \omega\,\wedge$$ covariant derivative

$$\omega \equiv d \log u$$ connection

⟹ $\varphi(z)$ and $\varphi(z) + \nabla_\omega \xi(z)$ are equivalent (in the sense of integration)

The equivalence classes form a vector space H^n_ω (the n-th twisted cohomology group)

$$\langle\varphi| \,:\, \varphi \sim \varphi + \nabla_\omega \xi$$

IBP reduction = vector decomposition

Cho, Matsumoto (1995)
Frellesvig et al. (2019-2020)
Weinzierl (2020)

$\dim(H^n_\omega) = \nu = \#$ of master integrals with a given ω

A basis with ν vectors $\{\langle e_1| , \langle e_2| , \ldots , \langle e_\nu|\}$

All vectors are linear combinations $\langle \varphi| = \sum_{i=1}^{\nu} c_i \langle e_i|$

To perform the vector decomposition, one introduces a dual space with elements

$$|\varphi_R\rangle \, : \, \varphi_R \sim \varphi_R + \nabla_{-\omega}\xi_R$$

The **intersection numbers** are "scalar-products" between vectors and dual-vectors

$$\langle \varphi_L|\varphi_R\rangle_\omega = \frac{1}{(2\pi i)^n}\int \iota_\omega(\varphi_L)\wedge\varphi_R = \frac{1}{(2\pi i)^n}\int \varphi_L \wedge \iota_{-\omega}(\varphi_R)$$

Canonical DEs for polylogarithmic integral families

Using these tools, we want to answer two questions

$$\mathrm{d}\vec{f}(\boldsymbol{x},\epsilon) = \epsilon\left(\sum_i \mathrm{dlog}(W_i(\boldsymbol{x}))\boldsymbol{A}_i\right)\vec{f}(\boldsymbol{x},\epsilon)$$

How do we find a canonical basis?

How do we construct the coefficient matrix (symbol letters and rational coefficients)?

Canonical bases from d-log integrands

Chen, Jiang, Xu, LLY (2020)
Chen, Jiang, Ma, Xu, LLY (2022)

See also: Dlapa et al. (2021)

The idea is simple: we look for integrands of the d-log form

$$\int_{\mathscr{C}} u(\mathbf{z}) \frac{Q\, dz_1 \wedge \cdots \wedge dz_n}{z_1^{a_1} \cdots z_n^{a_n} P_1^{b_1} \cdots P_m^{b_m}} = \int_{\mathscr{C}} \left[G(\mathbf{z})\right]^{\epsilon} \bigwedge_{j=1}^{n} d \log f_j(\mathbf{z})$$

Only simple poles for all variables

$$u(\mathbf{z}) = \left[P_1(\mathbf{z})\right]^{\gamma_1} \cdots \left[P_m(\mathbf{z})\right]^{\gamma_m}$$

Two simple building blocks

$$\mathrm{d}\log(z-c) = \frac{dz}{z-c}$$

$$\mathrm{d}\log(\tau[z,c;c_\pm]) = \frac{\sqrt{(c-c_+)(c-c_-)}dz}{(z-c)\sqrt{(z-c_+)(z-c_-)}}$$

$$\equiv \mathrm{d}\log \frac{\sqrt{c-c_+}\sqrt{z-c_-} + \sqrt{c-c_-}\sqrt{z-c_+}}{\sqrt{c-c_+}\sqrt{z-c_-} - \sqrt{c-c_-}\sqrt{z-c_+}}$$

Intersection numbers between d-log integrals are particularly simple!

Differential equations

Chen, Feng, LLY (2023)

We are now ready to derive the canonical DEs

$$\eta_{IJ} = \langle \varphi_I | \varphi_J \rangle$$

$$\langle \dot{\varphi}_I | \equiv \hat{\mathrm{d}} \langle \varphi_I | = \left(\hat{\mathrm{d}}\Omega\right)_{IJ} \langle \varphi_J |$$

$$\left(\hat{\mathrm{d}}\Omega\right)_{IK} = \langle \dot{\varphi}_I | \varphi_J \rangle \left(\eta^{-1}\right)_{JK}$$

All symbol letters can be read off from these intersection numbers

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Would like to have universal formulas

Intersection numbers from multivariate residues

Chestnov et al. (2022)

$$\langle \varphi_L | \varphi_R \rangle = \sum_{\boldsymbol{p}} \mathrm{Res}_{\boldsymbol{z}=\boldsymbol{p}} \left(\psi_L \hat{\varphi}_R \right)$$

$$\nabla_n \cdots \nabla_1 \psi_L \;=\; \varphi_L$$

The poles are determined by the *u*-function

$$u(z) = \left[P_1(z)\right]^{\gamma_1} \cdots \left[P_m(z)\right]^{\gamma_m}$$

Solving this higher partial DE is in general difficult, but simplified if φ_L is d-log

A complication: the poles can be non-factorized and/or degenerate, e.g.:

$$u = z_1^{\beta_1} z_2^{\beta_2} (z_1 + z_2)^{\beta_3}$$

Factorization transformations

Chen, Feng, LLY (2023)

It is possible to perform variable changes (in the spirit of sector decomposition) to factorize the non-factorized poles, such that

$$u(z)\Big|_{z\to p} \xrightarrow{\; z \to \boldsymbol{x}^{(\alpha)} \;} u(\boldsymbol{x}^{(\alpha)})\Big|_{\boldsymbol{x}^{(\alpha)}\to\boldsymbol{\rho}^{(\alpha)}} = \bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)}) \prod_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\gamma_i^{(\alpha)}}$$

Different (α) labels different variable changes

Non-vanishing

$$u = z_1^{\beta_1} z_2^{\beta_2} (z_1 + z_2)^{\beta_3} \xrightarrow{\; z_1 = x_1,\; z_2 = x_1(x_2 - 1) \;} u = x_1^{\beta_1+\beta_2+\beta_3} x_2^{\beta_3} (x_2 - 1)^{\beta_2}$$

One needs to iterate over different factorizations for complete result

Symbol letters from factorized poles

Chen, Feng, LLY (2023)

Since φ_I and φ_J have only simple poles, $\dot{\varphi}_I$ have at most double poles

In this case, the intersection numbers can be computed using simple formulas

$$\varphi^{(\boldsymbol{b})} = C^{(\boldsymbol{b})} \bigwedge_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{b_i} \mathrm{d}x_i^{(\alpha)} \qquad u(\boldsymbol{x}^{(\alpha)})\big|_{\boldsymbol{x}^{(\alpha)} \to \boldsymbol{\rho}^{(\alpha)}} = \bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)}) \prod_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\gamma_i^{(\alpha)}}$$

$$\langle \dot{\varphi}_I | \varphi_J \rangle \to \frac{C_I^{(-\boldsymbol{1})} C_J^{(-\boldsymbol{1})}}{\boldsymbol{\gamma}^{(\alpha)}} \hat{\mathrm{d}} \log \left(\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)}) \right)$$

One double pole

$$\langle \dot{\varphi}_I | \varphi_J \rangle \to -\frac{\gamma_k^{(\alpha)}}{\boldsymbol{\gamma}^{(\alpha)}} \hat{\mathrm{d}} \int C_I^{(\boldsymbol{b}_I)} C_J^{(\boldsymbol{b}_J)} \hat{\mathrm{d}} \rho_k^{(\alpha)}$$

Only simple poles

Selection rule: can be non-zero only if φ_I and φ_J share $(n-1)$-variable poles

Symbol letters from factorized poles

Chen, Feng, LLY (2023)

The integration can be recasted as d-logs by studying the univariate intersection numbers after taking the residues of the $(n-1)$-variable poles

$$-\frac{\gamma_k^{(\alpha)}}{\gamma^{(\alpha)}}\,\hat{\mathrm{d}}\int C_I^{(\boldsymbol{b}_I)} C_J^{(\boldsymbol{b}_J)}\,\hat{\mathrm{d}}\rho_k^{(\alpha)}$$

$$\langle\dot{\varphi}_I|\varphi_I\rangle = \sum_{\alpha\neq I}\frac{\gamma^{(\alpha)}}{\gamma^{(I)}}\,\hat{\mathrm{d}}\log(c_I-c_\alpha)+\eta_{II}\beta_0\,\hat{\mathrm{d}}\log P_0$$

$$\langle\dot{\varphi}_I|\varphi_J\rangle = -\hat{\mathrm{d}}\log(c_I-c_J)+\eta_{IJ}\beta_0\,\hat{\mathrm{d}}\log P_0\,,$$

$$\begin{aligned}\langle\dot{\varphi}_I|\varphi_I\rangle &= \frac{1}{\gamma^{(I)}}\,\hat{\mathrm{d}}\log(\bar{u}_I(c_I))-\hat{\mathrm{d}}\log(c_+-c_-)\\ &\quad+\hat{\mathrm{d}}\log(c_I-c_+)+\hat{\mathrm{d}}\log(c_I-c_-)\,,\\ \langle\dot{\varphi}_I|\varphi_J\rangle &= \langle\dot{\varphi}_J|\varphi_I\rangle = -\hat{\mathrm{d}}\log\tau[c_I,c_J;c_\pm]\,.\end{aligned}$$

Purely algebraic method to determine the symbol letters starting from a single u-function

A new algorithmic approach

Jiang, Liu, Xu, LLY (2024)

Problems of the previous approach

➤ Relying on the construction of d-log basis

➤ Not easy for algorithmic implementation (especially the factorization of poles)

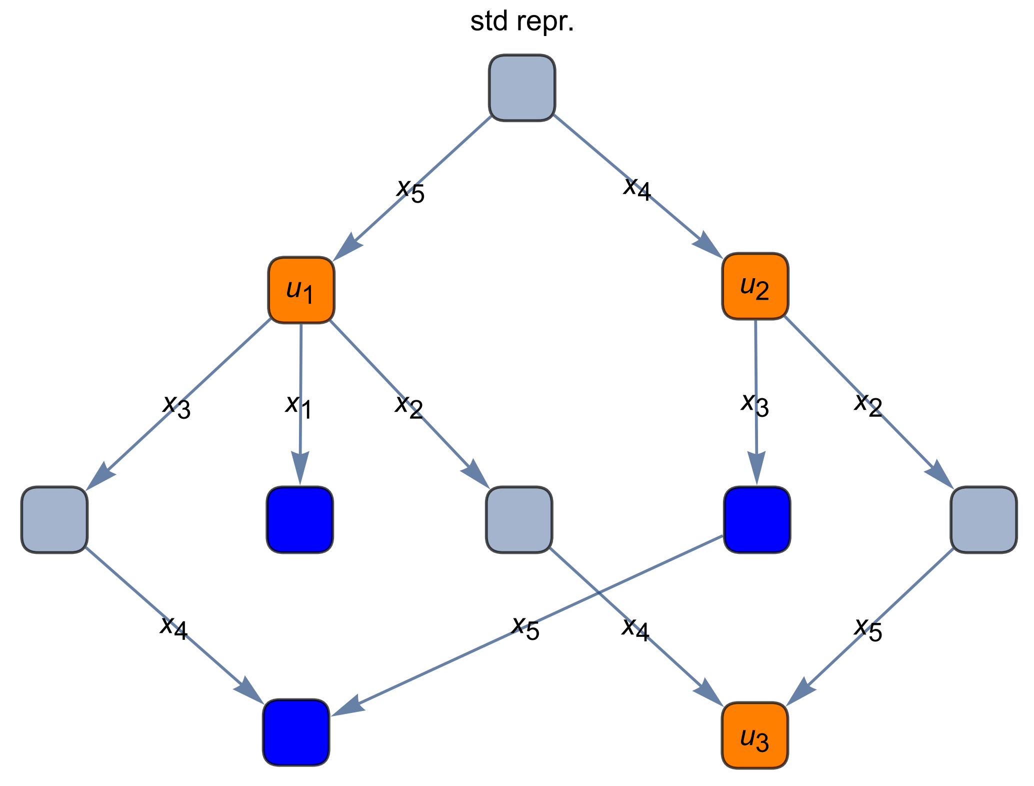
$$d\log \frac{G(\{q_1,q_2,\ldots,q_n,l\},\{q_1,q_2,\ldots,q_n,q_{n+1}\}) - \sqrt{-G(q_1,\ldots,q_n)G(q_1,\ldots,q_{n+1},l)}}{G(\{q_1,q_2,\ldots,q_n,l\},\{q_1,q_2,\ldots,q_n,q_{n+1}\}) + \sqrt{-G(q_1,\ldots,q_n)G(q_1,\ldots,q_{n+1},l)}}$$

$$d\log \frac{G(\{q_1,q_2,\ldots,q_n,l\},\{q_1,q_2,\ldots,q_n,q_{n+1}\}) - \sqrt{G(q_1,\ldots,q_{n+1})G(q_1,\ldots,q_n,l)}}{G(\{q_1,q_2,\ldots,q_n,l\},\{q_1,q_2,\ldots,q_n,q_{n+1}\}) + \sqrt{G(q_1,\ldots,q_{n+1})G(q_1,\ldots,q_n,l)}}.$$

But together with the generic one-loop results, it already hints at possible forms of symbol letters!

➤ They are written in terms of Gram determinants evaluated at certain singular points

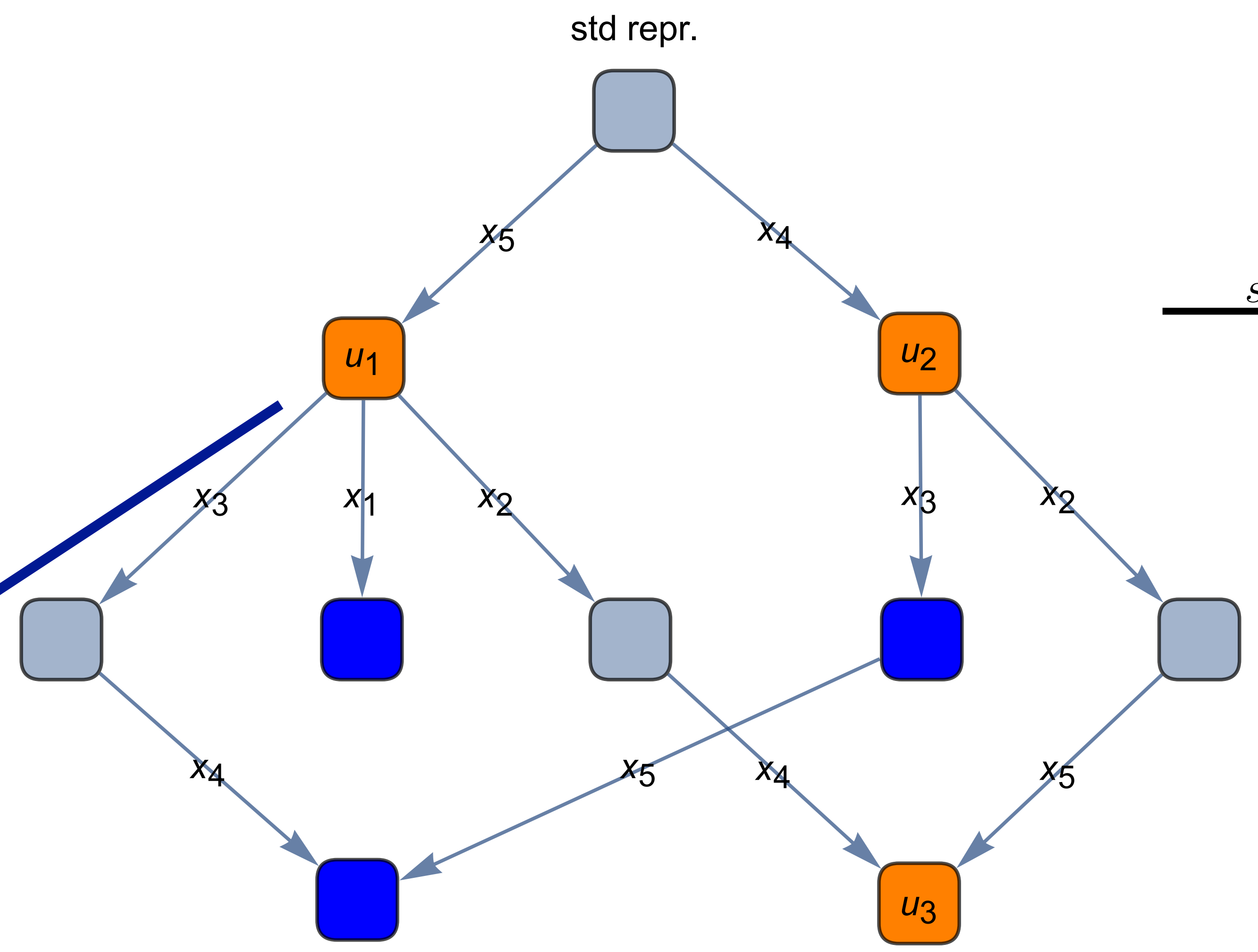
➤ These Gram determinants are connected in the recursive structure of Baikov representations

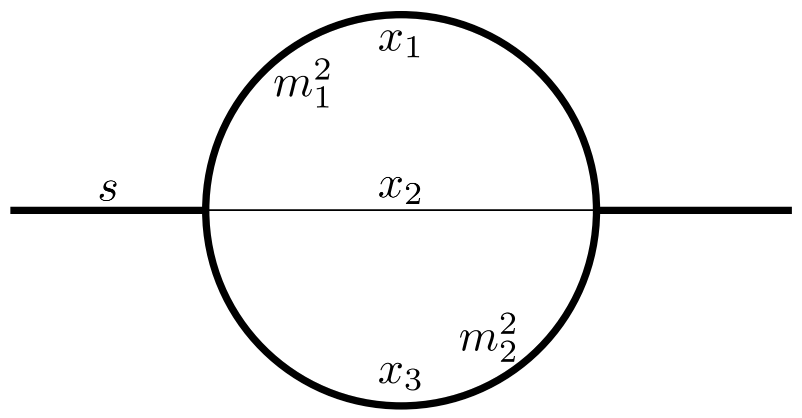


Identify the rational letters from leading singularities

Jiang, Liu, Xu, LLY (2024)

Constructing d-log integrands under maximal cut (much simpler than the full construction)





Or analyzing the singularities of the u-functions in the projective coordinates (particularly easy for algorithmic implementation)

$$\tilde{u}_1(x_4) = \left[\tilde{G}(p)\tilde{G}(l_2)\right]^{\epsilon}\left[\tilde{G}(l_1,l_2)\tilde{G}(l_2,p)\right]^{-1/2-\epsilon},$$

$$\tilde{G}(p) = s\,,\quad \tilde{G}(l_2,p) = -\lambda(x_4,s,m_2^2)/4\,,$$

$$\tilde{G}(l_2) = x_4\,,\quad \tilde{G}(l_1,l_2) = -(x_4-m_1^2)^2/4\,,$$

Singular points in the $[x_4 : x_0]$ space:

$[0 : 1]$, $[m_1^2,0]$, $[1,0]$

Scan all “minimal representations” → All rational letters

$s, m_1^2, m_2^2, \lambda(s, m_1^2, m_2^2)$

Search for irrational letters

Jiang, Liu, Xu, LLY (2024)

Look for irrational letters of the form

$$W(P,Q) = \frac{P+\sqrt{Q}}{P-\sqrt{Q}}$$

Combinations of Gram determinants

The Gram determinants that can be combined are not arbitrary!

- Constraints from recursive structure
- Constraints from d-log construction
- Relations among different determinants

$$B = G(\{\boldsymbol{k}, q_i\}, \{\boldsymbol{k}, q_j\})\,, \; A = G(\boldsymbol{k}, q_i)\,,$$
$$C = G(\boldsymbol{k}, q_j)\,, \; D = G(\boldsymbol{k})\,, \; E = G(\boldsymbol{k}, q_i, q_j)\,,$$

$$B^2 + DE = AC \quad \longrightarrow$$

$$\partial_B \log W(B, -DE)\Big|_{DE} = \frac{2\sqrt{-DE}}{-AC}\,,$$

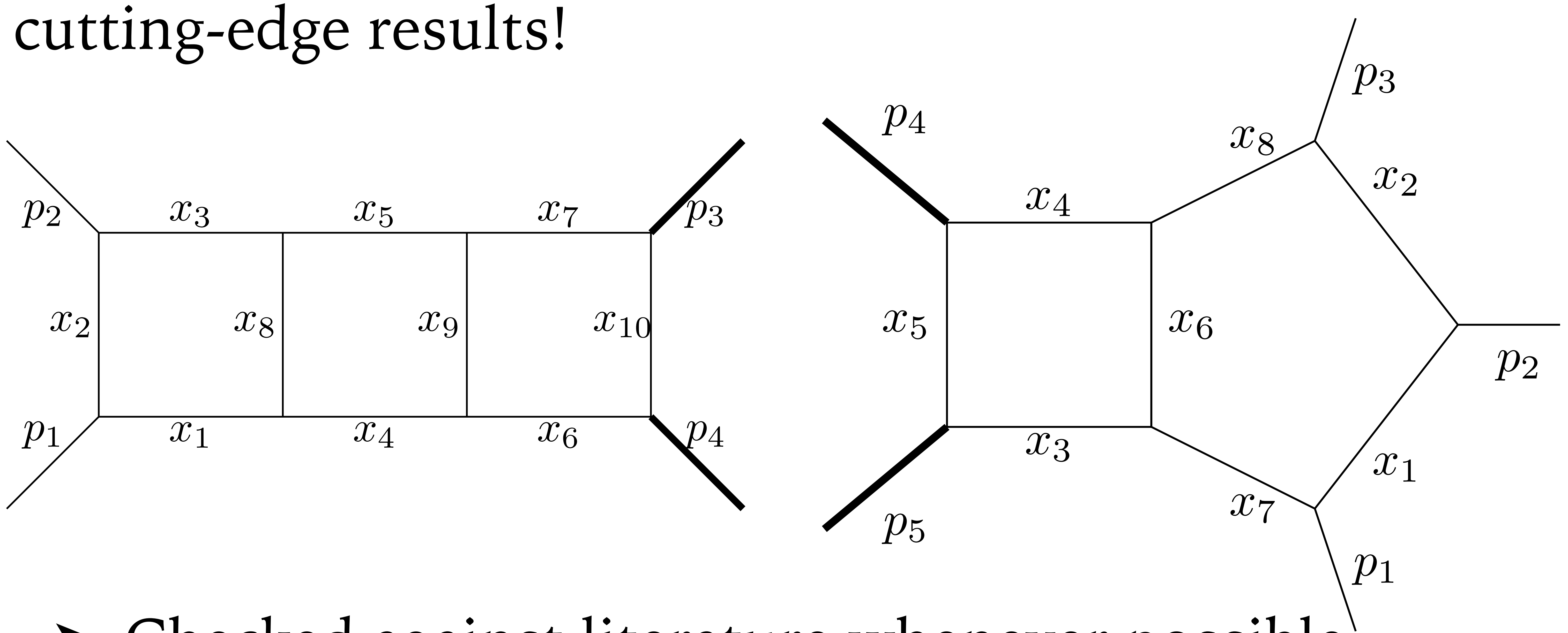
$$\partial_B \log W(B, AC)\Big|_{DE} = \frac{2}{\sqrt{AC}}\,,$$

Proof-of-concept implementation

https://github.com/windfolgen/Baikovletter

At the moment only for planar topologies

Not fully optimized, but already delivering many cutting-edge results!



- Checked against literature whenever possible
- Predict new results not available in literature
- Part of new results verified by bootstrapping the canonical DEs

```
rationalList = {m2, s, t, s + t, m2 - s, 4*m2 + s, 4*m2 + s + t, 4*m2 - s,
     4*m2 - t, s^2 - 4*m2*t + s*t, 4*m2*s + 4*m2*t - s*t,
     m2^2*s - 2*m2*s*t - 4*m2*t^2 + s*t^2}
squareRoots = {s*t*(-4*m2*s - 4*m2*t + s*t), s*(-4*m2 + s), (4*m2 - s)*s,
     s*(s + t)*(s^2 - 4*m2*t + s*t), s*(4*m2 + s), (4*m2 - t)*t,
     (s + t)*(4*m2 + s + t), s*(m2^2*s - 2*m2*s*t - 4*m2*t^2 + s*t^2), m2*s}
algebraicIndependent = {Log[-((-s + Sqrt[-((4*m2 - s)*s)])/
        (s + Sqrt[-((4*m2 - s)*s)]))],
     Log[-((-(s*t) + Sqrt[-(s*t*(4*m2*s + 4*m2*t - s*t))])/
        (s*t + Sqrt[-(s*t*(4*m2*s + 4*m2*t - s*t))]))],
     Log[-((-(s*(s + t)) + Sqrt[s*(s + t)*(s^2 - 4*m2*t + s*t)])/
        (s*(s + t) + Sqrt[s*(s + t)*(s^2 - 4*m2*t + s*t)]))],
     Log[-((2*m2*s + 4*m2*t - s*t + Sqrt[(4*m2 - s)*t*(4*m2*s + 4*m2*t -
            s*t)])/(-2*m2*s - 4*m2*t + s*t +
         Sqrt[(4*m2 - s)*t*(4*m2*s + 4*m2*t - s*t)]))],
     Log[-((-s^2 + 4*m2*t - s*t + Sqrt[(4*m2 - s)*(s + t)*(-s^2 + 4*m2*t -
            s*t)])/(s^2 - 4*m2*t + s*t + Sqrt[(4*m2 - s)*(s + t)*
           (-s^2 + 4*m2*t - s*t)]))],
     Log[-((-2*m2 + Sqrt[m2*(4*m2 - s)])/(2*m2 + Sqrt[m2*(4*m2 - s)]))],
     Log[-(((-I)*(m2*s + 4*m2*t - s*t) +
         I*Sqrt[-((4*m2 - s)*(m2^2*s - 2*m2*s*t - 4*m2*t^2 + s*t^2))])/
        (I*(m2*s + 4*m2*t - s*t) +
         I*Sqrt[-((4*m2 - s)*(m2^2*s - 2*m2*s*t - 4*m2*t^2 + s*t^2))]))],
     Log[-((-(t*(3*m2*s + 4*m2*t - s*t)) +
         Sqrt[-(t*(4*m2*s + 4*m2*t - s*t)*(m2^2*s - 2*m2*s*t - 4*m2*t^2 +
             s*t^2))])/(t*(3*m2*s + 4*m2*t - s*t) +
         Sqrt[-(t*(4*m2*s + 4*m2*t - s*t)*(m2^2*s - 2*m2*s*t - 4*m2*t^2 +
             s*t^2))]))],
     Log[-((I*(4*m2 - s)*t*(s + t) + I*Sqrt[t*(s + t)*(4*m2*s + 4*m2*t - s*t)*
            (-s^2 + 4*m2*t - s*t)])/((-I)*(4*m2 - s)*t*(s + t) +
         I*Sqrt[t*(s + t)*(4*m2*s + 4*m2*t - s*t)*(-s^2 + 4*m2*t - s*t)]))],
     Log[-((-2*m2*t + Sqrt[m2*t*(4*m2*s + 4*m2*t - s*t)])/
        (2*m2*t + Sqrt[m2*t*(4*m2*s + 4*m2*t - s*t)]))],
     Log[-((t + Sqrt[-((4*m2 - t)*t)])/(-t + Sqrt[-((4*m2 - t)*t)]))],
     Log[-((m2 + Sqrt[m2*s])/(-m2 + Sqrt[m2*s]))],
     Log[-((I*s + (2*I)*Sqrt[m2*s])/((-I)*s + (2*I)*Sqrt[m2*s]))],
     Log[-((2*m2 + s + Sqrt[s*(4*m2 + s)])/(-2*m2 - s +
         Sqrt[s*(4*m2 + s)]))],
     Log[-((I*(s + t) + I*Sqrt[(s + t)*(4*m2 + s + t)])/
        ((-I)*(s + t) + I*Sqrt[(s + t)*(4*m2 + s + t)]))],
     Log[-((s*(m2 + t) + Sqrt[s*(m2^2*s - 2*m2*s*t - 4*m2*t^2 + s*t^2)])/
        (-(s*(m2 + t)) + Sqrt[s*(m2^2*s - 2*m2*s*t - 4*m2*t^2 + s*t^2)]))],
     Log[-((s*(m2 - t) + Sqrt[s*(m2^2*s - 2*m2*s*t - 4*m2*t^2 + s*t^2)])/
        (-(s*(m2 - t)) + Sqrt[s*(m2^2*s - 2*m2*s*t - 4*m2*t^2 + s*t^2)]))],
     Log[-((4*m2*s + 2*m2*t - s*t + Sqrt[s*(4*m2 - t)*(4*m2*s + 4*m2*t -
            s*t)])/(-4*m2*s - 2*m2*t + s*t +
         Sqrt[s*(4*m2 - t)*(4*m2*s + 4*m2*t - s*t)]))],
     Log[-((s*(4*m2 + s + t) + Sqrt[-(s*(4*m2 + s + t)*(-s^2 + 4*m2*t -
             s*t))])/(-(s*(4*m2 + s + t)) +
         Sqrt[-(s*(4*m2 + s + t)*(-s^2 + 4*m2*t - s*t))]))],
     Log[-((t*(s + t) + Sqrt[-((4*m2 - t)*t*(s + t)*(4*m2 + s + t))])/
        (-(t*(s + t)) + Sqrt[-((4*m2 - t)*t*(s + t)*(4*m2 + s + t))]))],
     Log[-(((4*m2 + s)*t + Sqrt[-(s*(4*m2 + s)*(4*m2 - t)*t)])/
        (-((4*m2 + s)*t) + Sqrt[-(s*(4*m2 + s)*(4*m2 - t)*t)]))],
     Log[-((I*s + (2*I)*Sqrt[-(m2*s)])/((-I)*s + (2*I)*Sqrt[-(m2*s)]))],
     Log[-((-((4*m2 + s)*(s + t)) + Sqrt[s*(4*m2 + s)*(s + t)*
          (4*m2 + s + t)])/((4*m2 + s)*(s + t) +
         Sqrt[s*(4*m2 + s)*(s + t)*(4*m2 + s + t)]))]
```

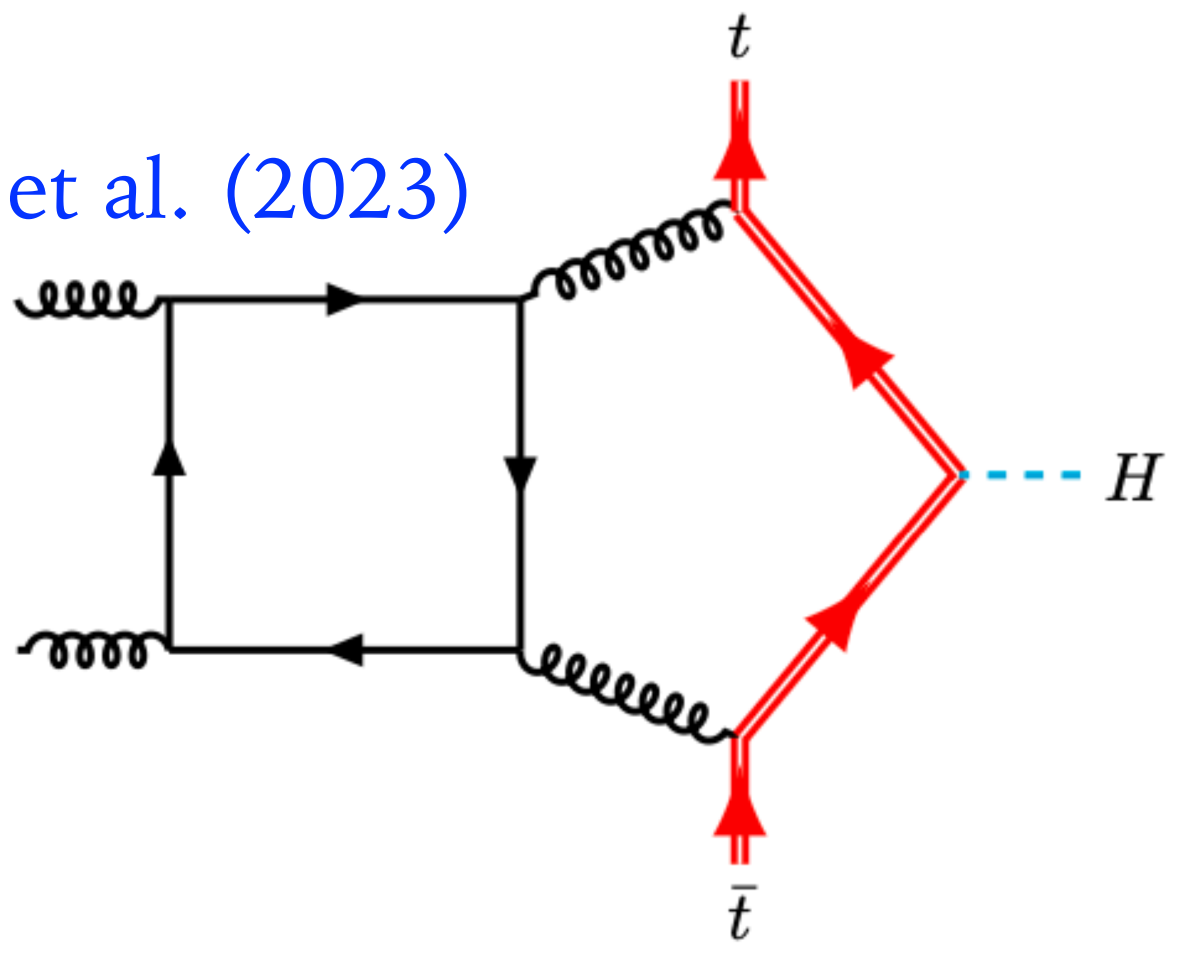
What's next?

- Obvious step: implementation for non-planar cases

- Stranger irrational letters (nested square roots)

$$\log \frac{R+\sqrt{P+\sqrt{Q}}}{R-\sqrt{P+\sqrt{Q}}}$$

Cordero et al. (2023)



- Beyond polylogarithms?

Beyond polylogarithms

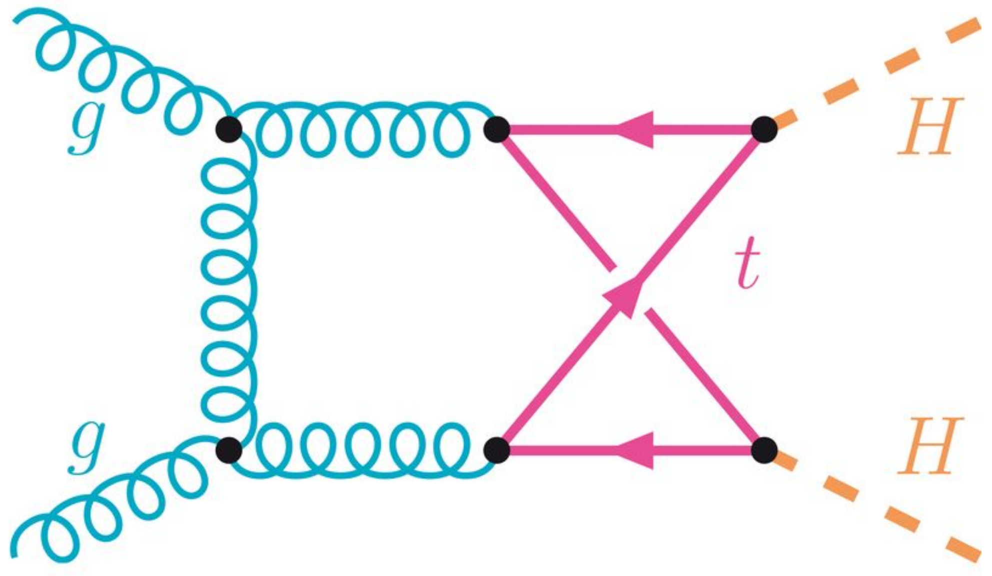
Elliptic integrals and iterated integrals over them

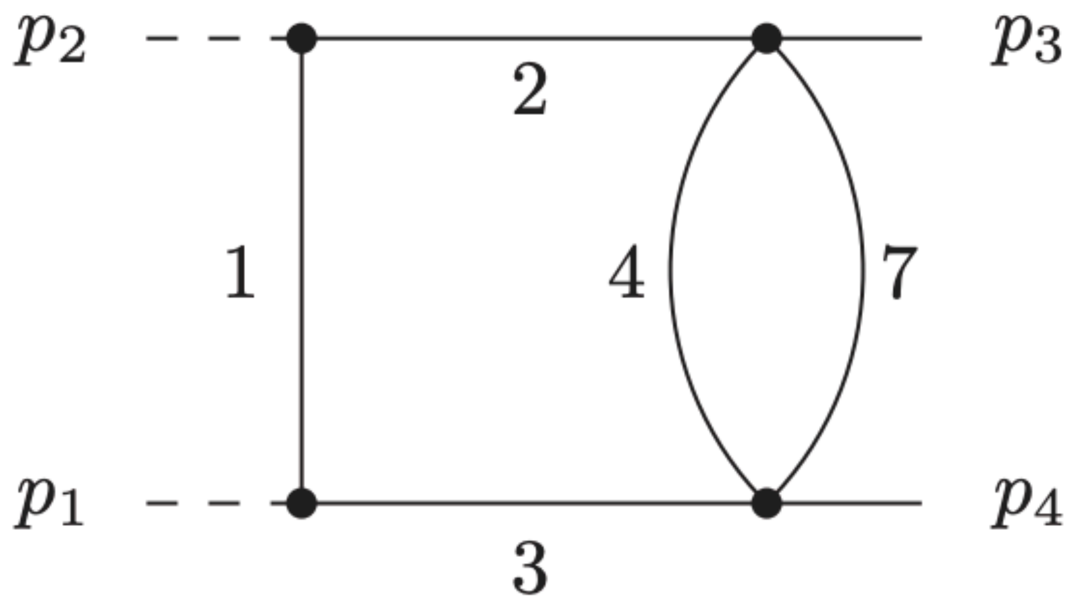
$$F(x;k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$E(x;k) = \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt$$

$$\Pi(n;\varphi\,|\,m) = \int_0^{\sin\varphi} \frac{1}{1-nt^2} \frac{dt}{\sqrt{(1-mt^2)(1-t^2)}}$$

Appearing in cutting-edge calculations





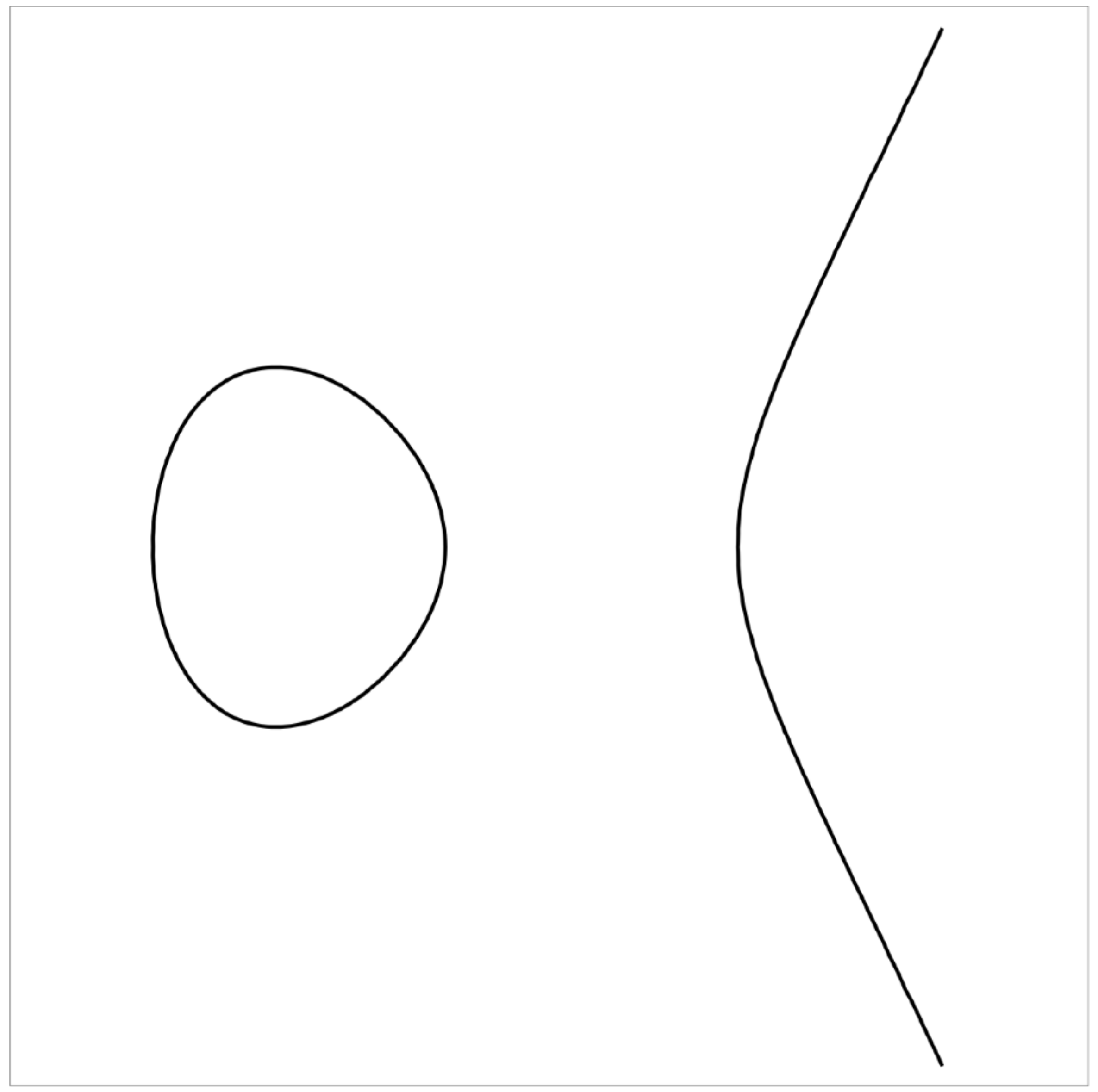
Many developments not covered here!

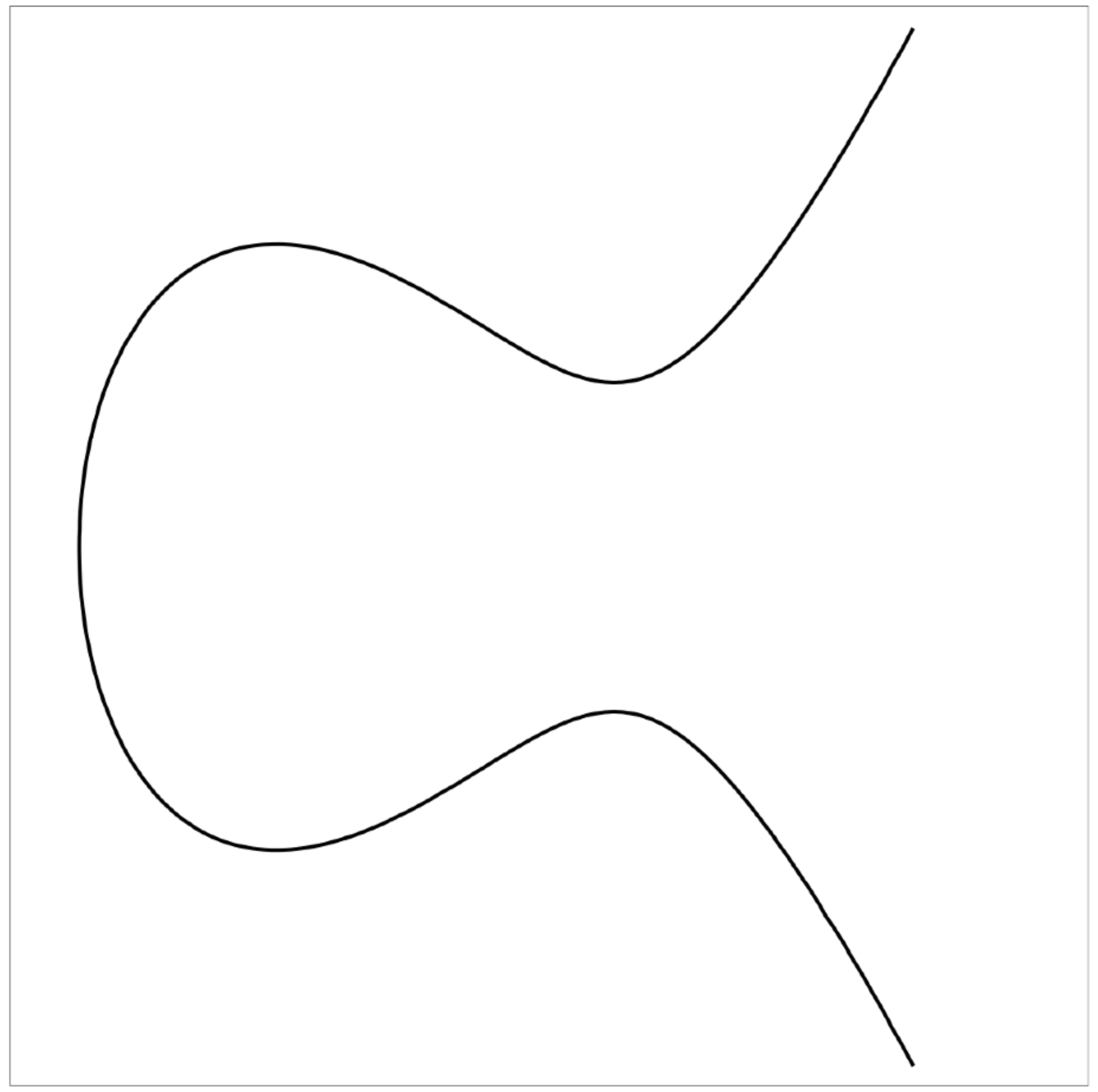
Elliptic integrals and elliptic curves

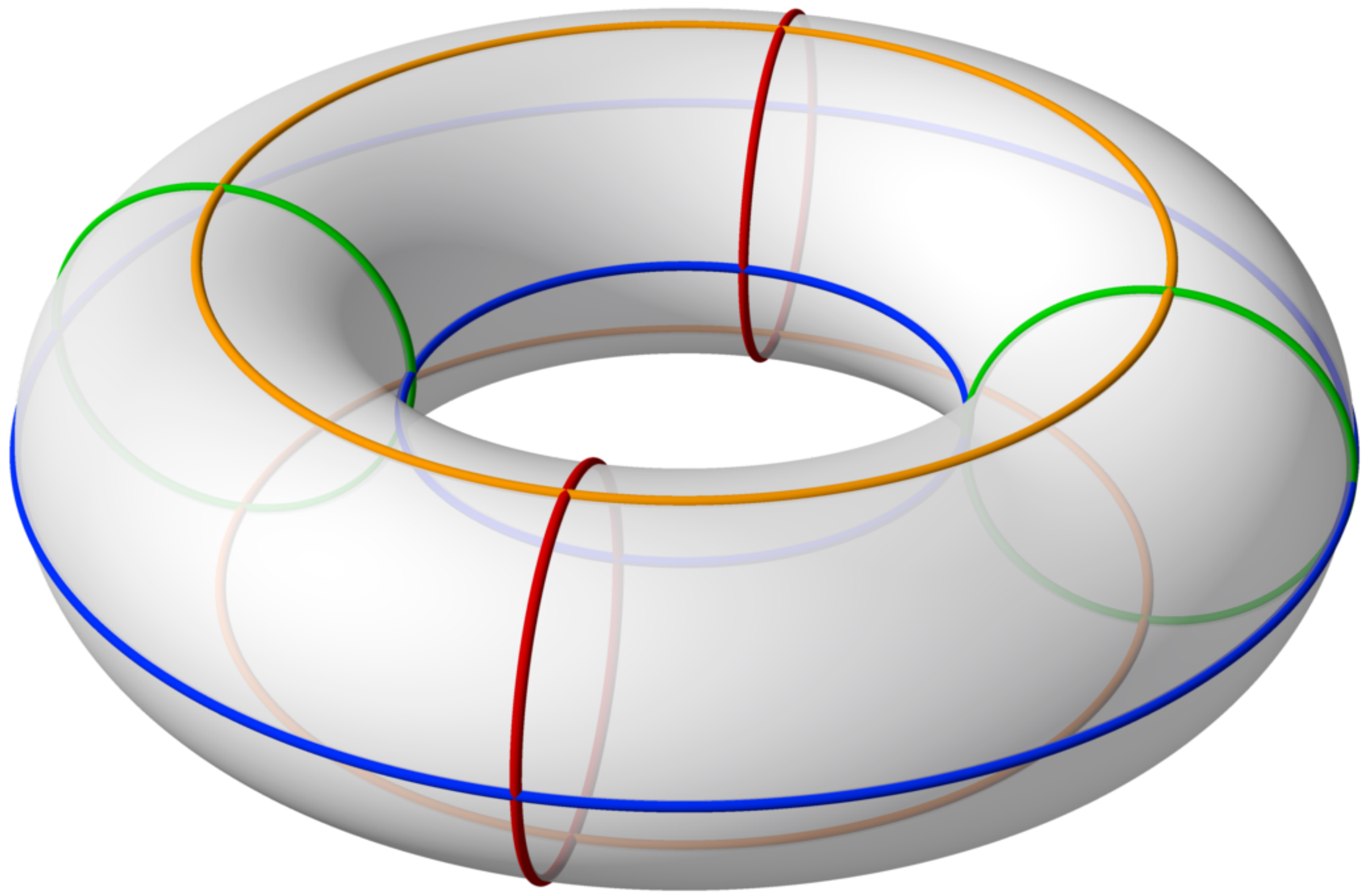
Functions can be categorized by the underlying geometry

- The geometric object underlying MPLs is a sphere
- The geometric object underlying iterated elliptic integrals is an elliptic curve (a torus)

$$y^2 = P(x) \qquad \text{(Degree-3 or 4 polynomial with distinct roots)}$$







Canonical DEs for elliptic integral families

Want to extend the concepts of canonical DEs to elliptic cases

$$\mathrm{d}\vec{f}(\boldsymbol{x},\epsilon) = \epsilon \left(\sum_i \mathrm{d}\alpha_i(\boldsymbol{x}) \boldsymbol{A}_i \right) \vec{f}(\boldsymbol{x},\epsilon)$$

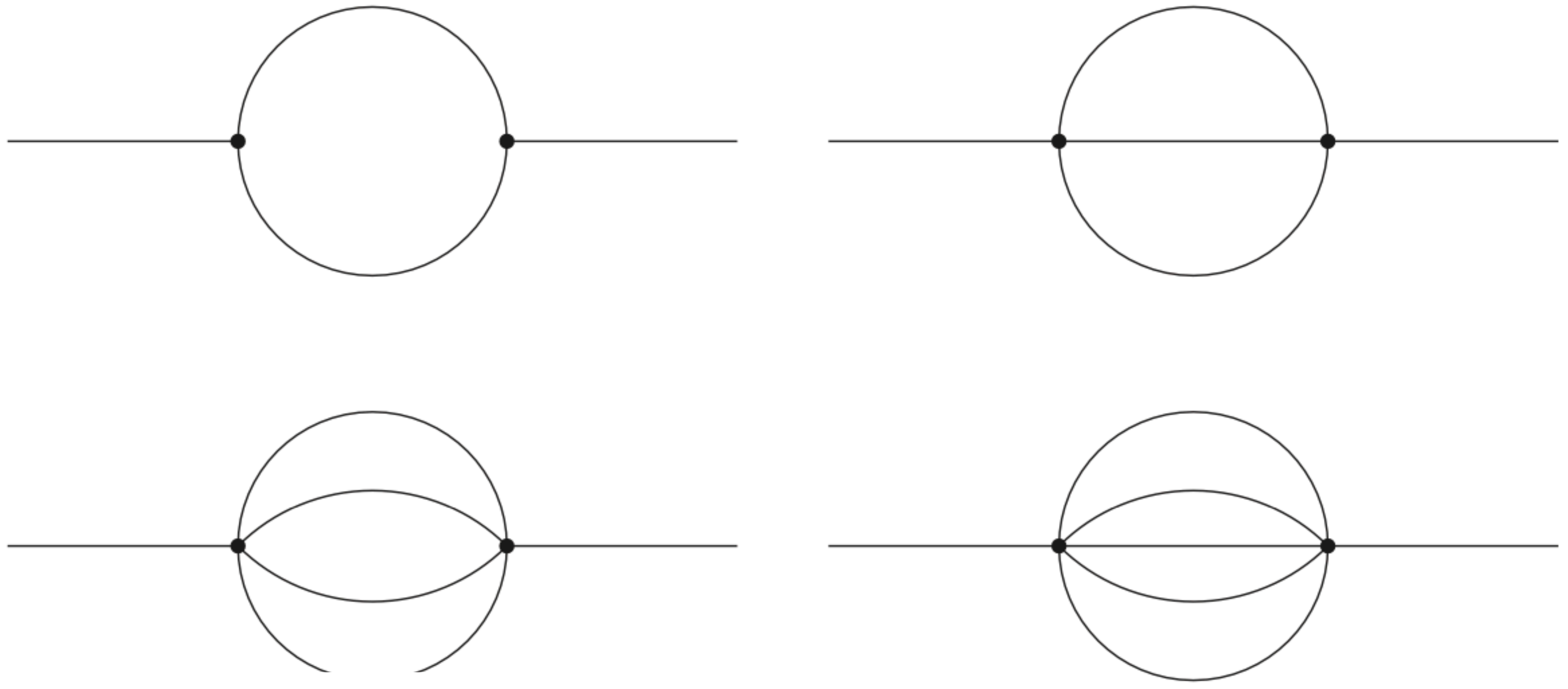
How to find a canonical basis?

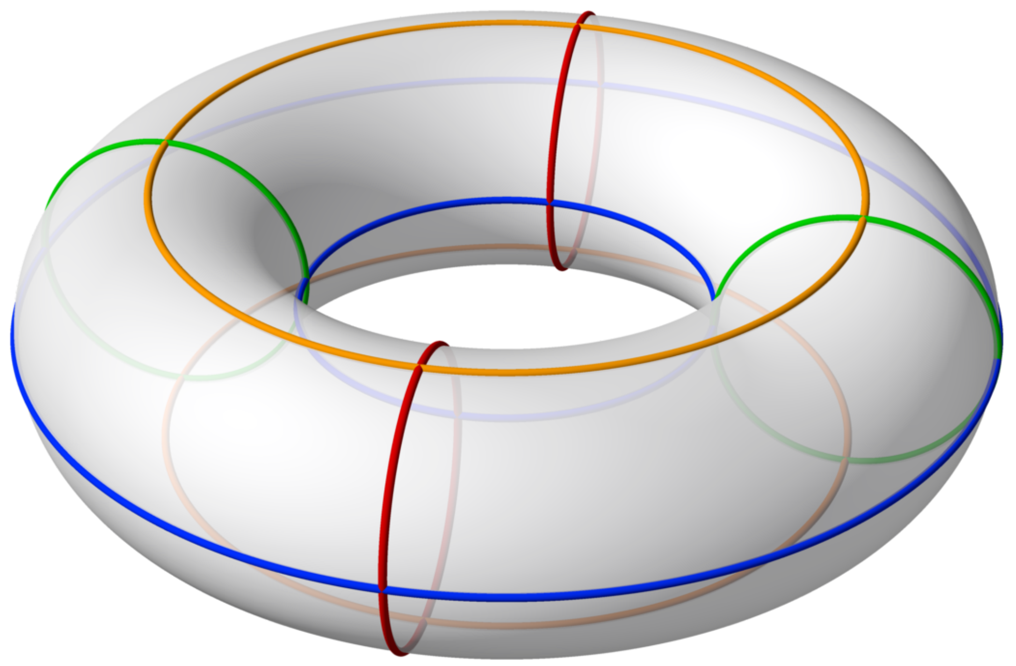
What are the corresponding symbol letters?

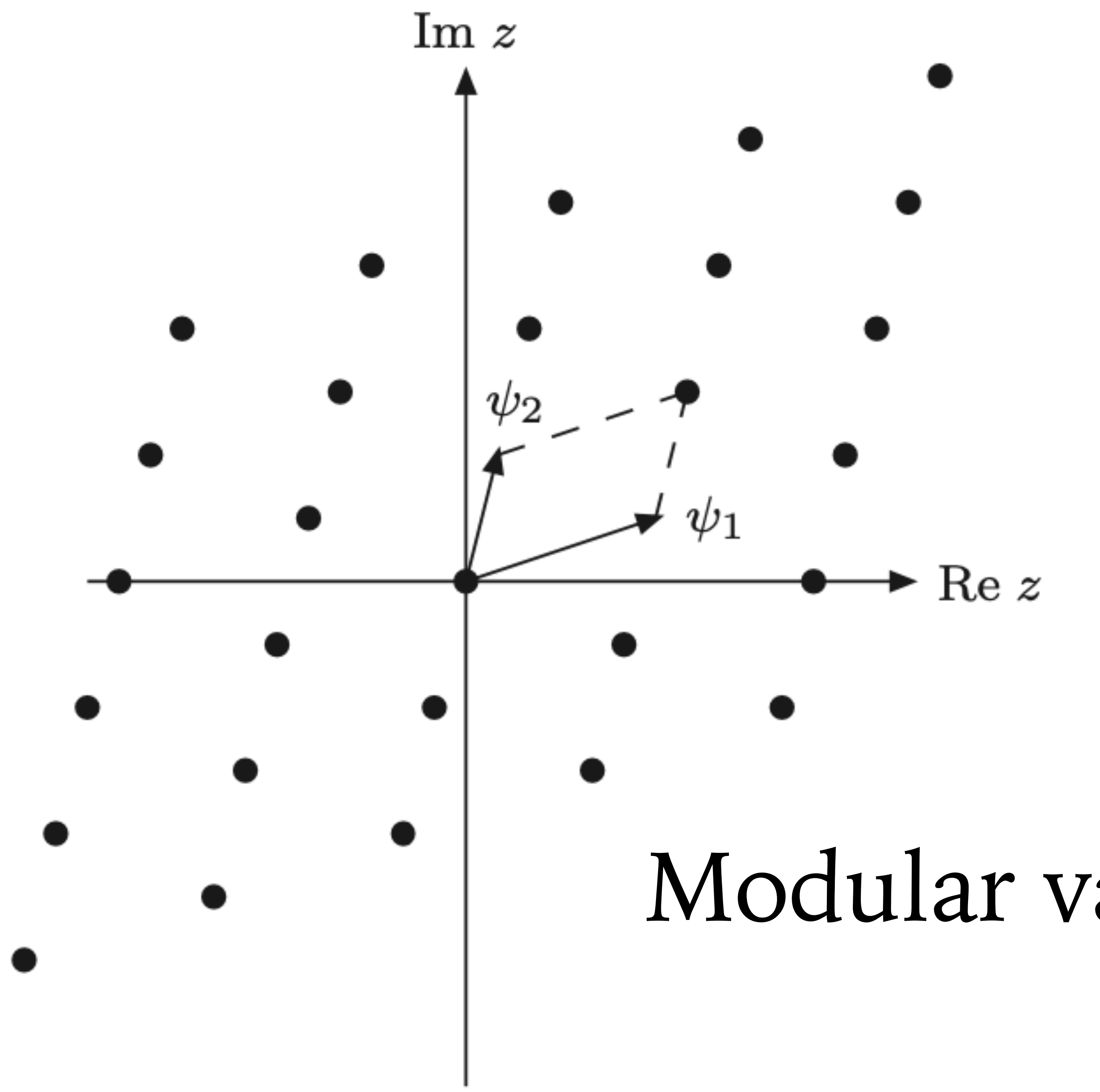
(No longer logarithms!)

Sunrise and Banana families

Pogel, Wang, Weinzierl (2022)







Lessons from equal-mass sunrise and banana families:

we should utilize modular transformations and modular forms associated with the elliptic curves

Modular variable $\tau = \dfrac{\psi_2}{\psi_1}$ Modular transformation $\tau \to \dfrac{a\tau + b}{c\tau + d}$

Sunrise and Banana families

Pogel, Wang, Weinzierl (2022)

The canonical DEs can be derived by analyzing the Picard-Fuchs operator, e.g.:

$$L_3^{(0)} = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2(x-4)} + \frac{3}{2(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2-68x+64}{x^2(x-4)(x-16)}\frac{d}{dx} + \frac{1}{x^2(x-16)}.$$

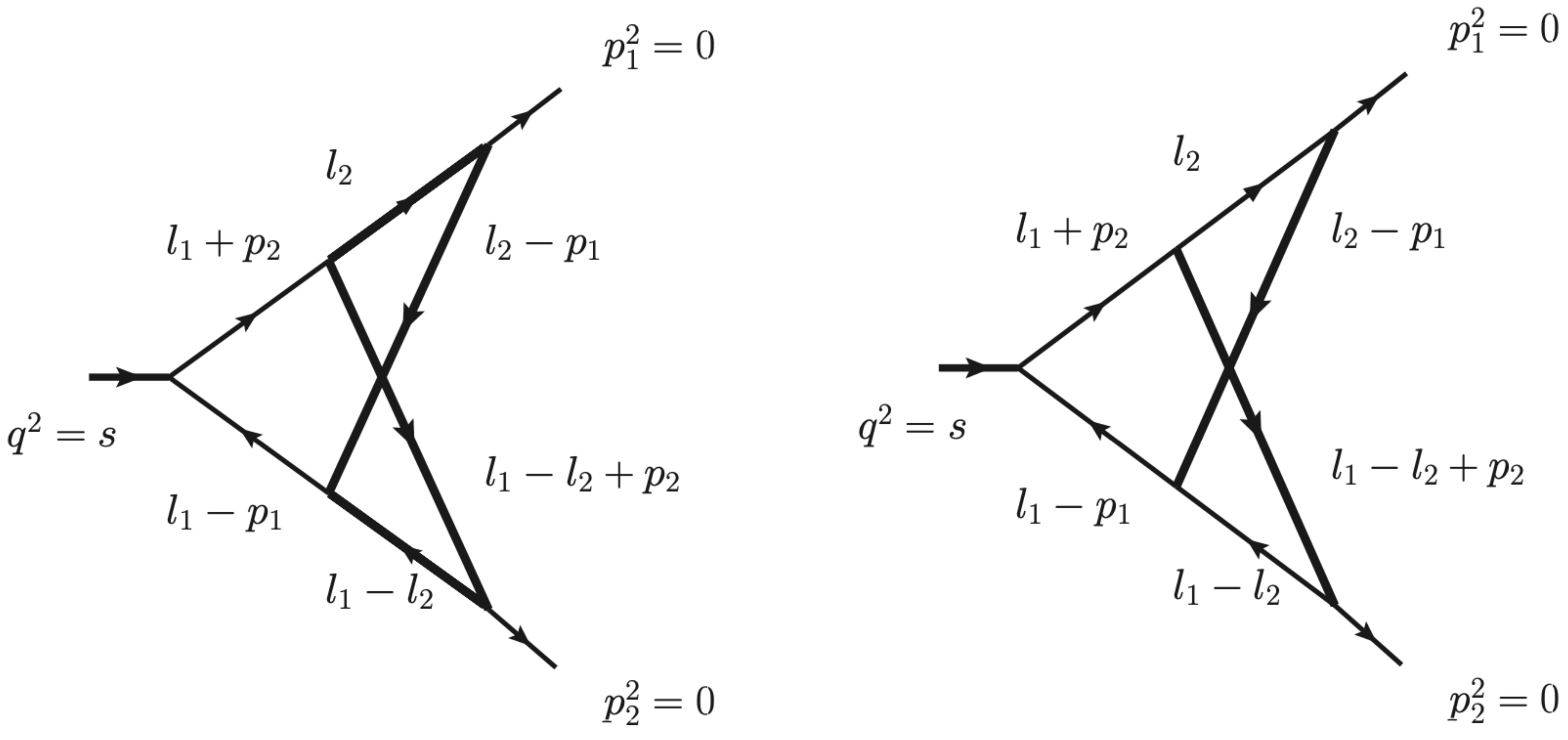
It turns out that the symbol letters can be expressed as modular forms, e.g.:

$$b_0 = \frac{2}{3}\sqrt{3}\frac{\eta(2\tau)^6\eta(3\tau)}{\eta(\tau)^3\eta(6\tau)^2}, \qquad b_1 = 6\sqrt{3}\frac{\eta(\tau)\eta(6\tau)^6}{\eta(2\tau)^2\eta(3\tau)^3}.$$

But this cannot be the whole story in more complicated cases!

Single parameter elliptic families with non-trivial sub-sectors

Jiang, Wang, LLY, Zhao (2023)



(a) (b)

$$y = -\frac{m^2}{s}$$

Appearing in, e.g., HH&ZH production and Higgs decays

Non-trivial sub-sectors: 2 top-sector MIs + 9 sub-sector MIs for family (a)

3 top-sector MIs + 15 sub-sector MIs for family (b)

Canonical DEs and solutions

Jiang, Wang, LLY, Zhao (2023)

Again possible to derive canonical DEs (including sub-sectors)

$$\frac{1}{2\pi i}\frac{d\vec{M}}{d\tau}=\varepsilon\begin{pmatrix}
\eta_{1,2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\eta_4 & \eta_{1,2} & 0 & 8\eta_3 & -12\rho & -28\eta_3 & 16\eta_3 & 10\eta_3 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\eta_{2,2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \eta_{2,2} & 0 & -\eta_{2,2} & \frac{\eta_{2,2}}{2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \eta_{2,2}+2\varpi_1 & 3\vartheta & 0 & 0 & 0 & -\vartheta & -\vartheta\\
0 & 0 & 0 & 0 & -\vartheta & -\eta_{2,2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\eta_{2,2} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\eta_{2,2} & -2\varphi & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\varphi & \eta_{2,2}+3\varpi_2 & 0 & -\varphi\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_{2,2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}\vec{M}$$

The symbol letters are no longer purely modular forms

Non-trivial contributions from singularities of sub-sectors (punctures on the torus)

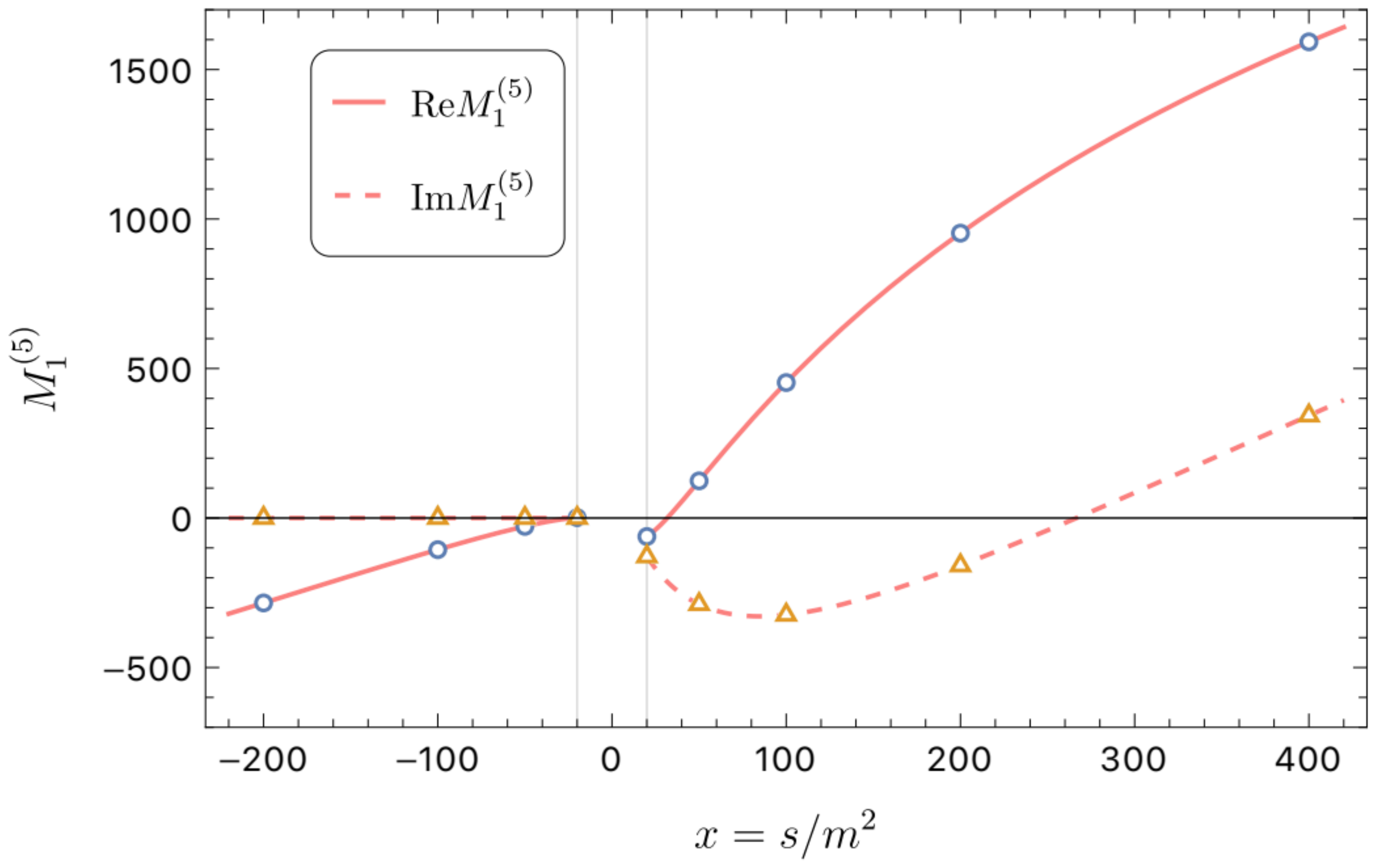
Open Question: can we construct these "elliptic symbol letters" algebraically?

Numerical evaluation

Jiang, Wang, LLY, Zhao (2023)

The iterated integrals can be evaluated using q-expansion $q = e^{2\pi i\tau}$

$$I(f_1, f_2, \ldots, f_n; \tau, \tau_0) = (2\pi i)^n \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \cdots \int_{\tau_0}^{\tau_{n-1}} d\tau_n \, f_1(\tau_1) f_2(\tau_2) \ldots f_n(\tau_n)$$



Needs to understand the analytic continuation and argument transformation properties of these iterated integrals!

Summary and outlook

- Towards building all symbol letters in a loop integral family automatically
  - Baikov representations + intersection theory
  - Algorithmic approach for planar cases (with proof-of-concept implementation)
  - Bootstrapping canonical DEs
- Extension to non-planar cases in progress
- Relation and combination with other approaches (Schubert, Landau, …)
- Applications in more situations…
- Extension to elliptic integrals and more complicated cases?

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Thank you!