3d N=2 from M-theory on CY4 and IIB brane box

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Classification of CFTs is an interesting but hard question
(1) 2d CFT: Virasoro algebra provides strong constraints, rational CFT
(2) For higher dimensional CFTs (e. g. $d \geq 3$), the full operator spectrum, OPEs ... are not known
Conformal Field Theories

- Classification of CFTs is an interesting but hard question
  1. 2d CFT: Virasoro algebra provides strong constraints, rational CFT
  2. For higher dimensional CFTs (e.g. $d \geq 3$), the full operator spectrum, OPEs ... are not known

- In the SCFT cases, partial classification comes from geometric constructions
  1. Superstring/M/F-theory on a non-compact space
  2. Dimensional reduction of 6d SCFTs on a compact space
  3. Worldvolume theory of brane objects in superstring/M/F-theory (AdS/CFT)
- **Superstring/M/F-theory** on a non-compact space, decouple gravity

String theory on
- **Compact X**
- **Quantum gravity**

String theory on
- **Non-compact X**
- **QFT**

Decompactify
String theory on Non-compact $X$

String theory on a singularity

Singular limit

QFT

CFT

- The CFT degree of freedoms are localized around the origin
5d SCFTs

(1) 11d M-theory on canonical threefold singularity

(Xie, Yau 15')(Apruzzi, Bhardwaj, Closset, Collinucci, De Marco, Del Zotto, Eckhard, Giacomelli, Heckman, Hubner, Jefferson, Katz, Kim, Lawrie, Lin, Morrison, Mu, Sangiovanni, Saxena, Schafer-Nameki, Tarazi, Tian, Vafa, Valandro, YNW, Zafrir, Zhang...).

(2) Brane web constructions in IIB superstring

Directly study the operator spectrum/OPE etc. Hard!

1. **Coulomb branch**: scalars $\phi^i$ in the vector multiplets have non-zero vev.

2. **Higgs branch**: scalars in the hypermultiplets have non-zero vev.
• M-theory on a resolved CY3 → CB physics, $U(1)^r +$ massive charged matter

CB U(1) gauge field $A$
from $C_3 = A \wedge \omega$

M2-brane wrapping 2-cycle $C$: BPS particle
$m \propto \text{Area}(C')$
Non-abelian and SCFT limit

- Non-abelian gauge theory description exists when the CY3 has a $\mathbb{P}^1$-fibration structure, e.g. the local $\mathbb{P}^1 \times \mathbb{P}^1$ gives 5d $SU(2)_0$ theory in the non-abelian limit.

- Similar picture in the IIB $(p, q)$ 5-brane web constructions!
What about $3d \mathcal{N} = 2$?

- Naturally, M-theory on local CY4 singularity $\rightarrow 3d \mathcal{N} = 2$ SCFT, because of the absence of geometric scale
- Build up geometric dictionary, investigate $3d \mathcal{N} = 2$ physics from M-theory on CY4 (Najjar, Tian, YNW 23').

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3d $\mathcal{N}=2$ from M-theory on CY4 and IIB brane box
3d $\mathcal{N} = 2$ basics

- Vector multiplet: $A_\mu, \lambda, \tilde{\lambda}$, real scalar $\sigma$
- Chiral multiplet: $\phi, \psi$ (same d.o.f. as 4d $\mathcal{N} = 1$ chiral multiplet)
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$$L = L_{YM} + L_{CS} + L_{matter} + L_{superpotential}$$ (1)
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- $[1/g_{YM}^2] \sim M^{-1}$, asymptotic freedom in UV, strongly coupled
  SCFT/gapped TQFT in IR
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- Real mass for a Dirac fermion $\psi$ in 3d: $im\bar{\psi}\psi$, $m \in \mathbb{R}$, odd under parity
- Integrate out "chiral" fermions $\rightarrow$ IR effective Chern-Simons terms
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- Integrate out “chiral” fermions $\rightarrow$ IR effective Chern-Simons terms
- Lots of IR dualities (Aharony, Hanany, Intriligator, Seiberg, Strassler 97')...
Resolved CY4 (CB)

- M-theory on resolved local CY4 $X_4$, e.g. local $D = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow 3d \mathcal{N} = 2$ U(1) gauge theory + massive matter fields

Non-compact 6-cycles
Flavor/Topological U(1)

U(1) gauge field from $C_3 = A \wedge \omega$

M2 wrapping 2-cycle: massive particles

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Uncharged sector

\[ C_3 = \sum_{i=1}^{r} A_i \wedge \omega_i^{(1,1)} + \sum_{\alpha=1}^{f} B_\alpha \wedge \omega_\alpha^{(1,1),F} \]  

(1) Dynamical gauge fields \( A_i \)
- Gauge rank \( r = b_6(X_4) \)
- \( \omega_i^{(1,1)} \) Poincaré dual to compact divisor (6-cycle) \( D_i \)

(2) Background gauge fields \( B_\alpha \) for geometric flavor symmetries
- Flavor rank \( f = b_2(X_4) - b_6(X_4) \)
- \( \omega_\alpha^{(1,1),F} \) Poincaré dual to non-compact divisor (6-cycle) \( S_\alpha \)
To compute volume of various cycles in $X_4$, we need the Kähler $(1,1)$-form

$$J(X_4) = \sum_{i=1}^{r} a_i \omega_i^{(1,1)} + \sum_{\alpha=1}^{f} b_\alpha \omega_{\alpha}^{(1,1), F}.$$  \hspace{1cm} (3)

It is Poincaré dual to

$$J^c(X_4) = \sum_{i=1}^{r} a_i D_i + \sum_{\alpha=1}^{f} b_\alpha S_\alpha.$$  \hspace{1cm} (4)
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(4)

(1) $a_i = \langle \sigma_i \rangle$: Coulomb branch parameters
(2) $b_\alpha = \langle \xi_\alpha \rangle$: vev for the real scalar in the background gauge field vector multiplet; real mass for flavor symmetry

Volume of 2-cycles $C$, 4-cycles $S$ and 6-cycles $D$ are computed as

$$V_C = \int_C J, \quad V_S = \frac{1}{2} \int_S J \wedge J, \quad V_D = \frac{1}{6} \int_D J \wedge J \wedge J.$$  

(5)
• $U(1)$ Gauge coupling $1/g^2$ given by what?
• Reduce the kinetic term in 11D SUGRA action on $X_4$ (leading term)

\[
\frac{1}{2} \int_{\mathbb{R}^{1,2} \times X_4} G_4 \wedge \ast G_4 = \frac{1}{2} \int_{\mathbb{R}^{1,2}} F \wedge \ast F \int_{X_4} \omega^{(1,1)} \wedge \ast \omega^{(1,1)} + (\ldots) \\
= \frac{1}{2g^2} \int_{\mathbb{R}^{1,2}} F \wedge \ast F + (\ldots)
\]

where

\[
\frac{1}{g^2} = \int_{X_4} \omega^{(1,1)} \wedge \ast \omega^{(1,1)} \\
= -\frac{1}{2} \int_{X_4} \omega^{(1,1)} \wedge \omega^{(1,1)} \wedge J \wedge J \\
= -\frac{1}{2} \int_{D} J \wedge J \\
= \text{Vol}(-K_D)
\]

• Volume of the anti-canonical divisor of $D!$
In the local $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ case, the compact divisor $D = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is toric, $\frac{1}{g^2}$ given by the sum of the volumes of all 4-cycles (walls)!
M2-brane wrapping modes

- BPS states from M2-brane wrapping \( \mathbb{P}^1 \) curves \( C \). Hint from 4d/3d F/M-duality (Beasley, Heckman, Vafa 08')(Intriligator, Jockers, Katz, Morrison, Plesser 12')(Jockers, Katz, Morrison, Plesser 16'). We first assume no \( G_4 \) flux
M2-brane wrapping modes

- BPS states from M2-brane wrapping $\mathbb{P}^1$ curves $C$. Hint from 4d/3d F/M-duality (Beasley, Heckman, Vafa 08') (Intriligator, Jockers, Katz, Morrison, Plesser 12') (Jockers, Katz, Morrison, Plesser 16'). We first assume no $G_4$ flux (1) $N_{C|X_4} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2)$, $C$ is locally a $\mathbb{P}^1$ fiber, moduli space is a 4-cycle $S$.

- Adiabatically, the zero modes on $C$ is the twisted reduction of 7d $\mathcal{N} = 1$ vector multiplet on $S$
- 1 vector multiplet + ($h^{0,1} + h^{0,2}$) vector-like pairs of chiral multiplets
(2) $N_{C|X_4} = \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $C$ is locally a $\mathbb{P}^1$ fiber, moduli space is a Riemann surface $\Sigma$.

- The zero modes on $C$ is the twisted reduction of 5d $\mathcal{N} = 1$ vector multiplet on $\Sigma$.
- BPS states come from zero modes of Dirac operators on $\Sigma \to$ vector-like pairs chiral multiplets.
- In particular, when $\Sigma = \mathbb{P}^1$, there is no zero mode and thus no BPS particles.
M2-brane wrapping modes

\[(2) \ N_{C|\mathbb{X}_4} = \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1), \ C \text{ is locally a } \mathbb{P}^1 \text{ fiber, moduli space is a Riemann surface } \Sigma.\]

- The zero modes on \( C \) is the twisted reduction of 5d \( \mathcal{N} = 1 \) vector multiplet on \( \Sigma \).
- BPS states come from zero modes of Dirac operators on \( \Sigma \rightarrow \) vector-like pairs chiral multiplets.
- In particular, when \( \Sigma = \mathbb{P}^1 \), there is no zero mode and thus no BPS particles.

- In general: mass of the BPS particle \( m \propto \text{Area}(C) \)
- Charge under Cartan: \( q = C \cdot D \)
- Charge under flavor Cartan \( q^F_i = C \cdot F_i \)
• Denote the non-compact divisors to be $S_1$, $S_2$, $S_3$, compact divisor is $D$ 

\[ C_a = D \cdot S_2 \cdot S_3, \quad C_b = D \cdot S_1 \cdot S_3, \quad C_c = D \cdot S_1 \cdot S_2 \quad (8) \]

• $C_a$, $C_b$, $C_c$ all have normal bundle $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2)$, moduli space $S = \mathbb{P}^1 \times \mathbb{P}^1$

• M2-brane wrapping mode: 1 vector multiplet

• $U(1)$ gauge charge $C_a \cdot D = C_b \cdot D = C_c \cdot D = -2$, hence one can choose $C_a$, $C_b$ or $C_c$ as gauge W-boson.
SU(2) limit

- In the limit of e.g. $\text{Area}(C_a) \to 0$, M2-brane wrapping $C_a$ becomes massless W-boson.

- $1/g^2 \sim \text{Vol}(S)$
- $SU(2)$ gauge theory + massive charged particle from M2-brane wrapping $C_b$ and $C_c$
- Interpreted as disorder operators! (Dyonic instanton in 5d $SU(2)_0$ theory on $S^2$)
SCFT limit

- Singular limit: all compact cycles shrink to zero volume, $1/g^2 \to 0$
- Absence of scale parameter $\to$ SCFT! $W = 0$
Shrinkability condition

- Geometric shrinkability condition for the existence of a 3d $\mathcal{N} = 2$ SCFT at singular point?

(1) At the singular limit when all compact cycles of $X^4$ shrinks to a point, still have $1/g^2 \alpha \rightarrow \infty$ for all non-compact divisors $S$ (counter example: local $D$ where $D$ is not weak-Fano).

(2) Exists strongly coupled limit $1/g^2 i \rightarrow 0$ for all $U(1)$ gauge groups only when the 4-cycles on all compact divisors $D_i$ in $X^4$ shrink to zero volume. (Counter example: 3d $\mathcal{N} = 4$ models such as local $T^2$.)
● Geometric **shrinkability** condition for the existence of a 3d $\mathcal{N} = 2$ SCFT at singular point?

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(2) Exists strongly coupled limit $1/g_i^2 \to 0$ for all $U(1)_i$ gauge groups only when the 4-cycles on all compact divisors $D_i$ in $X_4$ shrink to zero volume.

(Counter example: 3d $\mathcal{N} = 4$ models such as local $T^2$)
• In the singular limit of $X_4$, 3d $\mathcal{N}=2$ SCFT with non-abelian flavor symmetry enhancement $G_F$
• Read off from the CB picture from M-theory on resolved CY4
Flavor symmetry enhancement

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• Identify non-compact 6-cycles $F_i$ generating flavor Cartan $U(1)^f$
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• Read off from the CB picture from M-theory on resolved CY4

• Identify non-compact 6-cycles $F_i$ generating flavor Cartan $U(1)^f$

• Identify flavor W-bosons as M2 wrapping $C_i$.
  
  (1) Vector multiplet: $N_{C_i|X_4} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2)$
  
  (2) Charge under $U(1)^f$ forming the Cartan matrix of $G_F$
  
  (3) Neutral under $U(1)^r$ gauge symmetry
Flavor symmetry enhancement

- In the example of local \((\mathbb{P}^1)^3\), flavor Cartans

\[ F_1 = S_1 - S_2, \quad F_2 = S_2 - S_3. \]  

- Flavor W-bosons

\[ C_1 = D \cdot (S_1 - S_2) \cdot S_3, \quad C_2 = D \cdot (S_2 - S_3) \cdot S_1. \]
Flavor symmetry enhancement

- In the example of local \((\mathbb{P}^1)^3\), flavor Cartans

\[
F_1 = S_1 - S_2, \quad F_2 = S_2 - S_3. \tag{9}
\]

- Flavor W-bosons

\[
C_1 = D \cdot (S_1 - S_2) \cdot S_3, \quad C_2 = D \cdot (S_2 - S_3) \cdot S_1. \tag{10}
\]

\[
\begin{array}{c|cc}
  F_1 & F_2 \\
  \hline
  C_1 & -2 & 1 \quad , \quad \text{Exactly the Cartan matrix for } SU(3)! \\
  C_2 & 1 & -2
\end{array}
\tag{11}
\]
Flavor symmetry enhancement

- In the example of local \((\mathbb{P}^1)^3\), flavor Cartans

  \[ F_1 = S_1 - S_2, \quad F_2 = S_2 - S_3. \]  

- Flavor W-bosons

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- Exactly the Cartan matrix for \(SU(3)!\)  

- \(C_a, C_b\) and \(C_c\) form the 3 rep. of \(SU(3)!\)
Flavor W-boson being non-effective?

Similar to the 5d case, local $F_0 \approx$ local $F_2$ (Seiberg rank-1 $E_1$ theory with $G_F = SU(2)$)

Deformation $F_0 \to F_2$ gives the same SCFT!

Flavor W-boson is only an effective curve on $F_2$
• CY4 case, toric diagram from local $(\mathbb{P}^1)^3$:

• After the deformation, see the $SU(3)$ flavor symmetry explicitly
Flavor symmetry duality

- Sometimes, one can assign different sets of flavor W-bosons → Different non-abelian flavor symmetry enhancements
- Consider a 2d facet of a 3d toric diagram, with two $\mathbb{P}^1$-fibration structures

- $SU(3) \leftrightarrow SU(2)^2$ flavor symmetry duality! Phenomenon is not present in 5d
1-form symmetry

- 1-form global symmetry symmetry acting on Wilson loops of gauge theory (Gaiotto, Kapustin, Seiberg, Willett 14')
- Pure $d$-dim. $U(1)$ Maxwell theory has a $U(1)$ 1-form symmetry
1-form symmetry

- 1-form global symmetry acting on Wilson loops of gauge theory (Gaiotto, Kapustin, Seiberg, Willett 14')
- Pure $d$-dim. $U(1)$ Maxwell theory has a $U(1)$ 1-form symmetry
- $U(1)^r$ gauge theory w/ charged matter $\phi_i$ with charge $q_{ij}$ under $U(1)_j$, matter breaks the $U(1)^r$ 1-form symmetry to a subgroup $\Gamma$
- Compute Smith Normal Form $D$

$$D = \begin{pmatrix}
  l_1 & 0 & \cdots & 0 \\
  0 & l_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & l_r \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0
\end{pmatrix} = A\{q_{ij}\}B \quad (12)$$

$$\Gamma = \bigoplus_{i=1}^{r} (\mathbb{Z}/l_i\mathbb{Z}) \quad (13)$$
On the resolved $X_4$ CB,

1. $U(1)^r$ gauge fields from compact divisors $D_j$
2. Charged particles from M2-brane wrapping 2-cycles $C_i$

$q_{ij} = C_i \cdot D_j$, compute SNF, get 1-form symmetry
1-form symmetry

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- Local $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ example: all particles have even $U(1)$ charges $\rightarrow \Gamma = \mathbb{Z}_2$ 1-form symmetry!
1-form symmetry

- On the resolved $X_4$ CB,
  
  (1) $U(1)^r$ gauge fields from compact divisors $D_j$
  
  (2) Charged particles from M2-brane wrapping 2-cycles $C_i$

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- Local $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ example: all particles have even $U(1)$ charges $\rightarrow \Gamma = \mathbb{Z}_2$ 1-form symmetry!

- In the toric CY4 case, equivalent computation using SNF of list of toric rays ([Morrison, Schafer-Nameki, Willett 19’])
For M-theory/F-theory on CY4, (free) $G_4$ flux is usually a crucial ingredient:

$$G_4 + \frac{1}{2} c_2(X_4) \in H^4(X_4, \mathbb{Z})$$ (14)

In the non-compact CY4 case, $G_4$ should have compact support (dual to a compact 4-cycle) \cite{GukovVafaWitten:99}.

(1) Induce non-zero chirality of matter fields

Integrating out chiral matter $\rightarrow$ deep IR Chern-Simons term

$$S_{CS} = \frac{1}{4\pi} \sum_{i,j=1}^{k} k_{ij} A_i \wedge F_j.$$ (15)

$$k_{ij} = \int_{X_4} G_4 \wedge \omega((1,1) \wedge (1,1)) = G_{c4} \cdot D_i \cdot D_j.$$ (16)

(2) GVW superpotential, D-term superpotential

$$W_{GVW} = \int_{X_4} G_4 \wedge \Omega^4,$$

$$W_D = \int_{X_4} G_4 \wedge J \wedge J.$$ (17)
G\textsubscript{4} flux

- For M-theory/F-theory on CY4, (free) G\textsubscript{4} flux is usually a crucial ingredient

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1) Induce non-zero chirality of matter fields

- Integrating out chiral matter → deep IR Chern-Simons term

\[ S_{CS} = \frac{1}{4\pi} \int \sum_{i,j=1}^{r} k_{ij} A_i \wedge F_j. \] (15)

\[ k_{ij} = \int_{X_4} G_4 \wedge \omega_i^{(1,1)} \wedge \omega_j^{(1,1)} \] (16)

\[ = G_4^c \cdot D_i \cdot D_j. \]
For M-theory/F-theory on CY4, (free) $G_4$ flux is usually a crucial ingredient

$$G_4 + \frac{1}{2} c_2(X_4) \in H^4(X_4, \mathbb{Z})$$  \hspace{1cm} (14)

In the non-compact CY4 case, $G_4$ should have compact support (dual to a compact 4-cycle) (Gukov, Vafa, Witten, 99’)

1. Induce non-zero chirality of matter fields
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$$S_{CS} = \frac{1}{4\pi} \int \sum_{i,j=1}^{r} k_{ij} A_i \wedge F_j .$$  \hspace{1cm} (15)

$$k_{ij} = \int_{X_4} G_4 \wedge \omega_i^{(1,1)} \wedge \omega_j^{(1,1)}$$

$$= G_4^c \cdot D_i \cdot D_j .$$  \hspace{1cm} (16)

2. GVW superpotential, D-term superpotential

$$W_{GVW} = \int_{X_4} G_4 \wedge \Omega_4 , \quad W_D = \int_{X_4} G_4 \wedge J \wedge J .$$  \hspace{1cm} (17)
Now we consider the flux $G_4$ in the resolved $X_4$ (CB).

1. If $G_4 = 0$, there is no CS term in the deep IR.
2. If $G_4 \neq 0$, there is a CS term in the deep IR.

A flow diagram illustrates the RG flow from the deep UV to the IR, showing the transition from non-abelian gauge theory to the SCFT description at the singular limit. The $G_F$ enhancement is also indicated.
On the resolved $X_4$ (CB)

1. $G_4 = 0 \rightarrow$ no CS term in the deep IR
2. $G_4 \neq 0 \rightarrow$ CS term in the deep IR

Adding $G_4$ obstructs the singular limit of $X_4$! Because $G_4$ cannot pass through shrinking 4-cycles

To have the SCFT description at singular limit, $G_4 = 0$
A detailed computation of superpotential $W$ is still unknown, several sources

(1) Euclidean M5 brane wrapping compact 6-cycle $D$ with $h^{0,1}(D) = h^{0,2}(D) = h^{0,3}(D) = 0$ (Witten, 96’)

$$W_{EM5} = T(m_\alpha) e^{-V_D}.$$  \hfill (18)
Superpotential

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• Exists for any toric divisor!

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(2) Euclidean M2 brane wrapping rigid 3-cycles, absent in toric CY4.
(3) GVW superpotential w/ $G_4$ flux
• A detailed calculation of $W$ in the future?
Brane web picture in IIB

- In the case of toric CY4, a dual brane web description in IIB! (Leung, Vafa 97')

- First consider M-theory on $T^3$ (8, 9, 10) directions
- The toric CY4 is equivalent to the system of $(6 + 1)$-dim. KK7M monopoles

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• M-theory on $S_{10}^1 \rightarrow$ IIA

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Brane web picture in IIB

- M-theory on $S_{10}^1 \to$ IIA

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- T-duality along $S_9^1 \to$ IIB on $\tilde{S}_9^1$

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**Brane web picture in IIB**

- **M2-brane wrapping \( \mathbb{P}^1 \):** \( C_a, C_b, C_c \):

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- **In IIB description**

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Brane web picture in IIB

- Can be viewed as a web of \((p, q, r)\) 4-branes in 8d SUGRA (remove 8, 9 directions)! (Leung, Vafa 97')(Lu, Roy 98')
- \((p, q, r)\) transforms under \(SL(3, \mathbb{Z})\) (part of 8d U-duality)

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- M2-brane wrapping \(\mathbb{P}^1\): \(C_a, C_b, C_c\): \((1,0,0), (0,1,0), (0,0,1)\) strings!
- M2-brane wrapping 2-cycle in M-theory ↔ open string modes on 4-string junction!
• Generally for \((p, q, r)\)-string, mass of BPS open strings states

\[ m \sim \text{length} \times T_{(p, q, r)} \]  

(19)
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\[ m \sim \text{length} \times T_{(p,q,r)} \]  \hspace{1cm} (19)

- \(U(1)\) gauge field given by a linear combination of \(U(1)\)s of all finite 4-branes
- Electric charge of a string state \(Q_e = -N_b\), total \# of end points of the string
Flavor branes giving flavor symmetry

- Flavor branes in IIB: classified by exotic branes
- Flavor branes in 8D SUGRA: 5-brane objects

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$g_s^{-1}$

$g_s^{-2}$

$g_s^{-3}$
What’s next?

• Superpotential from geometry? Hard even for 3d $\mathcal{N} = 2 \ SU(2) + N_f F$!
• Higher derivative/quantum correction to the 11D SUGRA action, more precise formula for $1/g^2$
• Realize known 3d $\mathcal{N} = 2$ dualities, e. g. SQED-XYZ duality
• Relations to other 3d $\mathcal{N} = 2$ constructions, e. g. 6d (2,0) on 3-manifolds?
• 4d $\mathcal{N} = 1$ uplift in the elliptic cases
• Higgs branch?
• Detailed study of $\mathbb{C}^4/\Gamma$ orbifolds, 4d McKay correspondence
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Thank you for your attention!