Solving Scattering Equations

Bo Feng

East Joint Workshop on Fields and Strings 2016, May 27-31, 2016, Hefei, China

Based on the work 1509.04483, 1604.07314, 1605.06501 and unpublished results, with Rijun Huang, Junjie Rao, Yang-Hui He, Ming-xing Luo, Chuan-Jie Zhu, Carlos Cardona, Humberto Gomez,N. E. J. Bjerrum-Bohr, Jacob L. Bourjaily, Poul H. Damgaard

Contents

Part I: Backgrounds

In 2013, new formula for tree amplitudes of massless theories has been proposed by Cachazo, He and Yuan:

$$\mathcal{A}_n = \int \frac{\left(\prod_{i=1}^n dz_i\right)}{d\omega} \Omega(\mathcal{E}) \mathcal{I},$$

[Freddy Cachazo, Song He, Ellis Ye Yuan , 2013, 2014]

This formula contains three parts. For the first part:

- Integration variables are z_i's, i.e., locations of n external legs in sphere.
- The formula is invariant under the $SL(2, C)$ transformation $z \to \frac{az+b}{cz+d}$.
- The $d\omega$ is nothing, but the gauge volume and can be written as $d\omega = \frac{dz_r dz_s dz_t}{z_{rs} z_{st} z_{tr}}$.
- Dividing $d\omega$ will reduce integration to $(n-3)$ variables, i.e., three locations can be fixed by $SL(2, C)$ transformation.

The second part (measure part) is universal

$$\Omega(\mathcal{E}) \equiv \prod_{a}{}' \delta(\mathcal{E}_a) = z_{ij} z_{jk} z_{ki} \prod_{a \neq i,j,k} \delta(\mathcal{E}_a)$$

- Scattering equations are defined

$$\mathcal{E}_a \equiv \sum_{b \neq a} \frac{s_{ab}}{z_a - z_b} = 0, \qquad a = 1, 2, ..., n$$

- Only $(n-3)$ of them are independent by $SL(2, C)$ symmetry

$$\sum_a \mathcal{E}_a = 0, \quad \sum_a \mathcal{E}_a z_a = 0, \quad \sum_a \mathcal{E}_a z_a^2 = 0,$$

- $(n-3)$ integrations with $(n-3)$ delta-functions, so the integration becomes the sum over all solutions of scattering equations

$$\sum_{z \in \mathrm{Sol}} \frac{1}{\det'(\Phi)} \mathcal{I}(z)$$

where $\det'(\Phi)$ is the Jacobi coming from solve $\mathcal{E}_a$

$$\Phi_{ab} = \frac{\partial \mathcal{E}_a}{\partial z_b} = \begin{cases} \frac{s_{ab}}{z_{ab}^2} & a \neq b \\ -\sum_{c \neq a} \frac{s_{ac}}{z_{ac}^2} & a = b \end{cases},$$

The third part, i.e., CHY-integrand $\mathcal{I}(z)$, defines a particular theory.

- $SL(2,C)$ invariance require that under the transformation,

$$\mathcal{I}(z) \to \left(\prod_{i=1}^{n} \frac{(cz_i+d)^4}{(ad-bc)^2} \right) \mathcal{I}(z)\ .$$

  We will call $\mathcal{I}$ having weight four.

- To define proper CHY-integrand, let us define two building blocks. The first one is

$$\Sigma_\alpha(z) = \frac{1}{z_{\alpha(1)\alpha(2)} \cdots z_{\alpha(n-1)\alpha(n)}}, \quad \alpha \in S_n/Z_n$$

  which has weight two.

- The second building block is

$$E(\epsilon, k, z) = (\mathrm{Pf}'\Psi(k, \epsilon, z))$$

where $z_{ij} \equiv z_i - z_j$ and the $2n \times 2n$ matrix Ψ is given by

$$\Psi_{ab} = \begin{cases} \frac{k_a \dot{k}_b}{z_{ab}}, & a \neq b \\ 0, & a = b \end{cases}, \quad \Psi_{a+n,b+n} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{z_{ab}}, & a \neq b \\ 0, & a = b \end{cases}$$
$$\Psi_{a+n,b} = \begin{cases} \frac{k_a \cdot \epsilon_b}{z_{ab}}, & a \neq b \\ -sum_{c \neq a} \Psi_{c+n,b}, & a = b \end{cases} \tag{1}$$

and **the reduced Pfaffian** $\mathrm{Pf}'\Psi \equiv \frac{(-)^{i+j}}{z_{ij}} \mathrm{Pf}\Psi^{ij}_{ij}$ with $1 \leq i < j \leq n$, which has weight two.

Having pieces with weight two, we can multiply them to get weight four integrand:

- Bi-adjoint ϕ^3 scalar theory with ordering (α, β):

$$\mathcal{I}(z) = \Sigma_\alpha(z)\Sigma_b(z)$$

- Partial ordered YM-theory

$$\mathcal{I}(z) = \Sigma_\alpha(z)E(\epsilon, k, z)$$

- Gravity theory

$$\mathcal{I}(z) = E(\epsilon, k, z)E(\epsilon', k, z)$$

- Based on above two blocks, there are several manipulations on them to get more theories.

[Freddy Cachazo, Song He, Ellis Ye Yuan , 2014]

With above discussions, it is understandable that solving scattering equations will be a crucial part of the whole algorithm!
However, directly solving scattering equations is not an easy task!

- With proper transformation, we can change scattering equations to polynomial equations of multiple variables

$$0 = h_m \equiv \sum_{S \in A, |S|=m} k_S^2 z_S \; , \quad 2 \leq m \leq n-2 \; ,$$

where the sum is over all $\frac{n!}{(n-m)!m!}$ subsets S of $A = \{1, 2, ..., n\}$ with exactly m elements and $k_S = \sum_{b \in S} k_b$ and $z_S = \prod_{b \in S} z_b$.

[Dolan, Goddard, 2014]

- After gauge fixing, they define a **zero-dimensional ideal** in the polynomial ring in $n-3$ variables. Then, using the standard **Bézout's theorem**, the number of points in this ideal (solutions of the scattering equation) is $(n-3)!$.

- One can see this fact by noticing that after using the elimination theorem, it is reduced to a polynomial of a single variables degree $\prod_{m=1}^{n-3} \deg(\widetilde{h}_m) = (n-3)!$ with $\deg(\widetilde{h}_m) = m$.

- With this picture, it is easy to see that when $n \geq 6$, solving it analytically is almost impossible!

Furthermore, there are a few facts which are not so obvious by above direct method:

- Although each solution is very complicated, when we sum them together, we do get rational function of $k_i \cdot k_j$.
- Different CHY-integrands may give the same final answer. How to understand it? It is equivalent to determine when a CHY-integrand gives zero contribution.
- How to see the pole structure from CHY-integrand?

In this talk, we will concentrate on solving scattering equations and understanding above problems.

Part II: Companion Matrix

The first important observation is that what we really want is not individual solutions, but the sum over solutions ! Thus if there is an algorithm to make the sum without solving, it will be perfect. One of such algorithms is the companion matrix

[B. Sturmfels, https://math.berkeley.edu/ bernd/cbms.pdf]

The key is to realize that polynomial scattering equations have define an idea in ring $R = C[z_1, ..., z_{n-3}]$. Thus we have transformed the problem to computational algebraic geometry!

The mathematical statement is following:

- Suppose a Gröbner basis for I has been found for some appropriate monomial ordering and B is an associated monomial basis for I, which can be seen as a vector space of dimension d. Then the multiplication map by the coordinate variable x_i

$$\begin{aligned} & R/I \longrightarrow R/I \qquad (2) \\ T_i : \quad & f \longrightarrow x_i f \qquad (3) \end{aligned}$$

  is an endomorphism of quotient rings.
- In the basis B of monomials, this is a $d \times d$ matrix and is called a **companion matrix**

- Clearly, $\{T_i\}$ all mutually commute and thus can be simultaneously diagonalized.
- We have the following

Theorem (Stickelberger)

The complex roots z_i of I are the vectors of simultaneous eigenvalues of the companion matrices $T_{i=1,\ldots,n}$, i.e., the corresponding zero dimensional variety consists of the points:

$$\mathcal{V}(I) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \exists v \in \mathbb{C}^n \forall i : T_i v = \lambda_i v\} \ .$$

In particular, we have the following important consequence: Our desired quantity

$$\sum_{j=1}^{N} r(z_j) = \mathrm{Tr}[r\,(T_1, \ldots, T_n)]$$

where the evaluation of the rational function r on the matrices T_i is without ambiguity since they mutually commute.

We remark that because r is rational, whenever the companion matrices appear in the denominator, they are to be understood as the inverse matrix.

Example:

- The idea is

$$I := \langle xy - z, yz - x, zx - y\rangle \subset R = \mathbb{C}[x, y, z] .$$

- The expressions needed to be evaluated are:

$$p(x, y, z) = 3x^3y + xyz, \quad Q(x, y, z) = \frac{3x^3y + xyz}{2xy^2 + 4z^2 + 1} .$$

- In the lex ordering of $x \prec y \prec z$, the Gröbner basis and the monomial basis are, respectively,

$$\begin{aligned} GB(I) &= \left\langle z^3 - z, yz^2 - y, y^2 - z^2, x - yz \right\rangle \\ B &= \{1, y, yz, z, z^2\} . \end{aligned}$$

- Therefore, we have that, in the quotient ring R/I,

$$\begin{aligned} x.B &= \{yz, z, z^2, y, yz\}\,, \quad y.B = \{y, z^2, z, yz, y\}\,, \\ z.B &= \{z, yz, y, z^2, z\} \end{aligned}$$

so that

$$\begin{aligned} T_x &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad T_y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \\ T_z &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Thus

$$
\begin{aligned}
p &= \mathrm{Tr}\left(3T_x^3T_y + T_xT_yT_z\right) = 4, \\
Q &= \mathrm{Tr}\left((3T_x^3T_y + T_xT_yT_z)(2T_xT_y^2 + 4T_z^2 + I)^{-1}\right) = \frac{20}{21}
\end{aligned}
$$

Remarks:

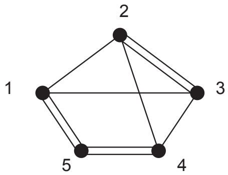
- We do not need to solve the equations
- At every step, it is rational expression
- Finding the companion matrix is not so easy
- It is not clear how the pole appear
- Similar algebraic approach (Bezoutian matrix method) has been proposed

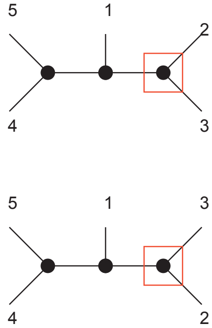
[Sogaard and Zhang, 2015]

Part III: Feynman rule

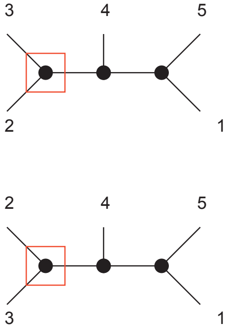
- It is obviously desirable to have a method giving wanted result without much calculations
- A first hint is given by the conjectured bi-adjoint ϕ^3 theory. [Freddy Cachazo, Song He, Ellis Ye Yuan , 2013]
- It is observed for CHY-integrand $\mathcal{I}(z) = \Sigma_\alpha(z)\Sigma_\beta$, the result is given by sum of Feynman diagrams consistent with two color orderings

Example $\frac{1}{(12345)(13245)}$ with $(a_1 \ldots a_m) = z_{a_1 a_2} \ldots z_{a_m a_1}$





$\frac{1}{s_{23}s_{45}}$



$\frac{1}{s_{23}s_{51}}$

Above conjecture has motivated following careful analysis:

[Baadsgaard, Bjerrum-Bohr, Bourjaily and Damgaard, 2015]

- Pole s_A with subset A appears when corresponding $z_{i\in A}$'s approach each other
- Under this limit, with rescaling $z_{i\in A} = \epsilon x_i$, the integration can be split to

$$\oint d\epsilon \epsilon^{\chi(A)-1} \oint dz_{i\notin A} \oint dx_{i\in A} \ldots .$$

- For simple case $\chi(A) = 0$, the integration of $d\epsilon$ can be carried out and the expression is reduce to

$$\left(\oint dz_{i\notin A} \ldots \right) \frac{1}{s_A} \left(\oint dx_{i\in A} \ldots \right)$$

which has a very clear picture of Feynman diagram

Above analysis leads to following important observation:

- There is an index characterizing the degree of pole for given subset A

$$\chi(A) := \mathbb{L}[A] - 2(|A| - 1)$$

$\mathbb{L}[A]$ be the number (more accurately it is the difference of number between solid and dashed lines) of lines connecting these nodes inside A and $|A|$ is the number of nodes.

- It has nonzero contribution when and only when $\chi(A) \geq 0$ and the pole will be

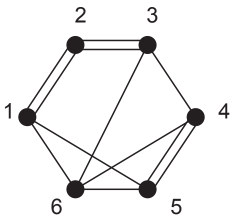
$$\frac{1}{s_A^{\chi(A)+1}}$$

The integration algorithm:

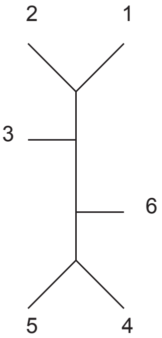
- Find all subsets A with $\chi(A) \geq 0$
- compatible condition for two subsets A_1, A_2: they are compatible if one subset is completely contained inside another subset or the intersection of two subsets is empty.
- Find all maximum compatible combinations, i.e., the combination of subsets with largest number such that each pair in the combination is compatible. For each maximum combination with m subsets, it gives nonzero contribution when and only when $m = n - 3$.

- Each combination giving nonzero contribution will correspond a (generalized) Feynman diagram with only cubic vertexes
- Now the key is how to read out expressions of Feynman diagrams?
- For simple pole, the rule is nothing, but the scalar propagator $\frac{1}{s_A}$!

Example of 6-point

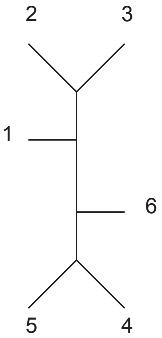


{1, 2}, {2, 3}, {4, 5}, {1, 2, 3}



{1, 2} + {4, 5} + {1, 2, 3}

$$\frac{1}{s_{12}s_{123}s_{45}}$$



{2, 3} + {4, 5} + {1, 2, 3}

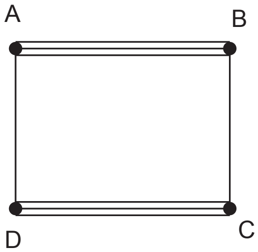
$$\frac{1}{s_{23}s_{123}s_{45}}$$

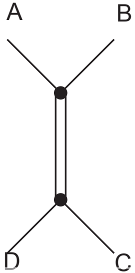
What is the Feynman rule for higher order poles?

- Key: The derivative property of residue of higher order pole makes it quasi-local, i.e., it depends not only the total momentum flow through the propagator, but also momentum configuration at the four corners.
- Simple pole is completely local.

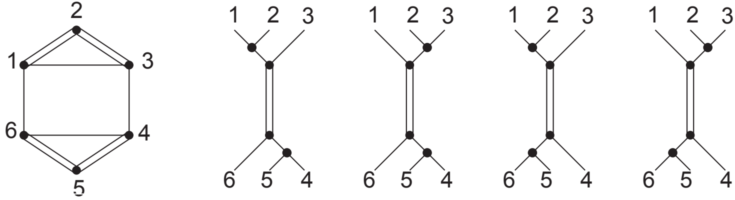
Feynman rule for single double pole:

$$\mathcal{R}_I[P_A, P_B, P_C, P_D] = \frac{2P_A P_C + 2P_B P_D}{2s_{AB}^2} ,$$





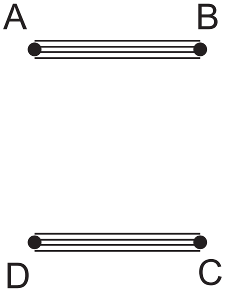
Example: Pole subsets $\underline{\{1,2,3\}}, \{1,2\}, \{2,3\}, \{4,5\}, \{5,6\}$

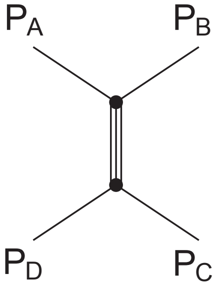


$$\frac{2p_{12}p_{45}+2p_3p_6}{2s_{123}^2 s_{12}s_{45}}+\frac{2p_{12}p_4+2p_3p_{56}}{2s_{123}^2 s_{12}s_{56}}$$
$$+\frac{2p_1p_{45}+2p_{23}p_6}{2s_{123}^2 s_{23}s_{45}}+\frac{2p_1p_4+2p_{23}p_{56}}{2s_{123}^2 s_{23}s_{56}}$$

Feynman rule for single triple pole

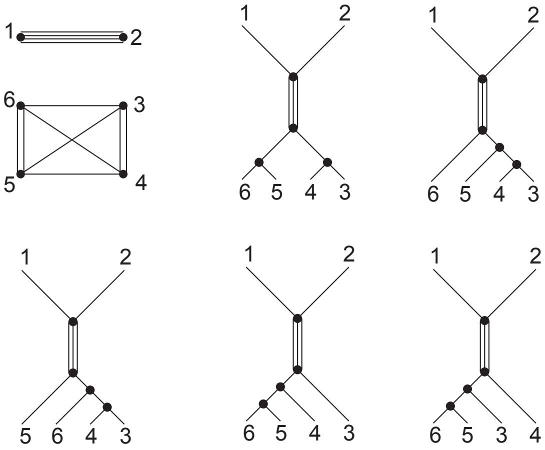
$$\mathcal{R}_{II}[P_A, P_B, P_C, P_D] = \frac{(2P_AP_C)(2P_AP_D)}{4s_{AB}^3} + \frac{(2P_BP_C)(2P_BP_D) + (2P_CP_A)(2P_CP_B) + (2P_DP_A)(2P_DP_B)}{4s_{AB}^3} - \frac{(P_A^2 - P_B^2)^2 + (P_C^2 - P_D^2)^2}{4s_{AB}^3} + \frac{2}{9}\frac{(P_A^2 + P_B^2)(P_C^2 + P_D^2)}{4s_{AB}^3} .$$





Pole subsets

$\{\underline{\underline{1,2}}\}, \{3,4\}, \{5,6\}, \{3,4,5\}, \{3,4,6\}, \{3,5,6\}, \{4,5,6\}$

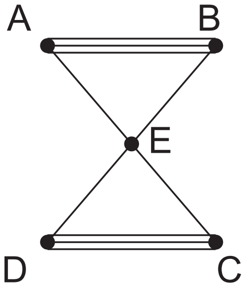


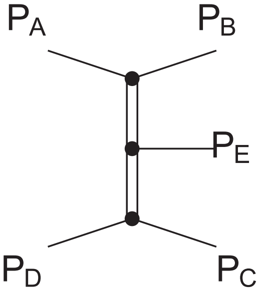
$$
-\frac{s_{15}}{s_{12}^2 s_{34} s_{56}} - \frac{s_{23} s_{15}}{s_{12}^3 s_{34} s_{56}} - \frac{s_{24} s_{15}}{s_{12}^3 s_{34} s_{56}} + \frac{1}{s_{12}^2 s_{56}} - \frac{s_{16}}{s_{12}^2 s_{34} s_{56}} - \frac{s_{16} s_{23}}{s_{12}^3 s_{34} s_{56}}
$$

$$
-\frac{s_{16} s_{24}}{s_{12}^3 s_{34} s_{56}} + \frac{s_{13} s_{23}}{s_{12}^3 s_{56} s_{456}} + \frac{s_{16} s_{26}}{s_{12}^3 s_{34} s_{345}} + \frac{s_{15} s_{25}}{s_{12}^3 s_{34} s_{346}} + \frac{s_{14} s_{24}}{s_{12}^3 s_{56} s_{356}} .
$$

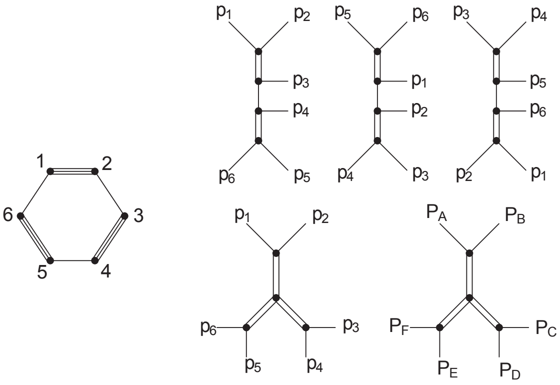
Feynman rule for duplex-double pole:

$$\mathcal{R}_{III}[P_A, P_B, P_E, P_C, P_D] = \frac{(2P_AP_D)(2P_BP_C) - (2P_AP_C)(2P_BP_D)}{s_{AB}^2 s_{CD}^2} - \frac{(P_E^2)(2P_AP_D + 2P_BP_C - 2P_AP_C - 2P_BP_D)}{4s_{AB}^2 s_{CD}^2}.$$





Feynman rule for triplex-double pole:



Part IV: Cross Ratio Identities

- Although Feynman rule method is very convenient, deriving rule for higher order poles is not systematic. A systematic way is to use cross ratio identities.
- With a little algebra, scattering equations can be rewritten as

$$1 = -\sum_{b \neq a,q,p} \frac{s_{ab}}{s_{aq}} \frac{z_{aq} z_{bp}}{z_{ab} z_{qp}}$$

- Let us use it for 4-point example

$$\begin{aligned} I_{4;A} &= \frac{1}{z_{12}^3 z_{23} z_{34}^3 z_{41}} \left(-\frac{s_{13}}{s_{12}} \frac{z_{12} z_{34}}{z_{13} z_{24}} \right) = -\frac{s_{13}}{s_{12}} \left(\frac{1}{z_{12}^2 z_{13} z_{34}^2 z_{24}} \right) \\ &\to -\frac{s_{13}}{s_{12}} \times \frac{1}{s_{12}} \end{aligned}$$

Now we see the systematic algorithm:

- Constructing the cross ratio identities for arbitrary pole

$$-s_A = -s_{\overline{A}} = \sum_{i \in S/\{p\}} \sum_{j \in \overline{S}/\{q\}} s_{ij} \frac{z_{ip} z_{jq}}{z_{ij} z_{pq}}$$

- Each multiplication of the identity will reduce the power of pole by one. Iterating enough times to reach simple pole.

Monodromy relation:

- For color ordered Yang-Mills amplitude, there are various relations, such as KK-relation and BCJ-relation
- KK-relation: [Kleiss, Kujif, 1989]

$$A_n(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in OP(\{\alpha\},\{\beta^T\})} A_n(1,\sigma,n)\ .$$

  where sum is over partial ordering.
- BCJ-relation: [Bern, Carraso, Johansson, 2008]

$$A_n(1,2,\{\alpha\},3,\{\beta\}) = \sum_{\sigma_i\in POP} A_n(1,2,3,\sigma_i)\mathcal{F},$$
$$\alpha = \{4,5,...,m\}$$
$$\beta = \{m+1,m+2,...,n\}$$

- These two relations can be understood in string theory as the real and imaginary parts of monodromy relation

[Bjerrum-Bohr, Damgaard, Vanhove, 2009]

[Stieberger, 2009]

- BCJ relation can be reduced to following fundamental BCJ relation

$$
\begin{aligned}
0 &= s_{21}A(1234...n) + ... + (\sum_{i=1}^{k} s_{2i})A(13...k2(k+1)...n) \\
&+ ... + (\sum_{i=1}^{n-1} s_{2i})A(13...k2(k+1)...n)
\end{aligned}
$$

- It is amazing to notice that if we exchange $A(12...(n-1)n) \to (12...(n-1)n) \equiv \frac{1}{z_{13}z_{34}z_{45}...z_{(n-1)n}z_{n1}}$, similar BCJ-relation holds

$$
\begin{aligned}
0 &= s_{21}(1234...n) + ... + (\sum_{i=1}^{k} s_{2i})(13...k2(k+1)...n) \\
&+ ... + (\sum_{i=1}^{n-1} s_{2i})(13...k2(k+1)...n)
\end{aligned}
$$

if z_i's are solutions of scattering equations.

The simple proof using scattering equations:

- After removing same factors, this identity becomes

$$0 = \left(s_{21} \frac{z_{13}}{z_{12} z_{23}} + \sum_{k=3}^{n-1} \Big(\sum_{i=1}^{k} s_{2i} \Big) \frac{z_{k(k+1)}}{z_{k2} z_{2(k+1)}} \right) .$$

- Collecting coefficients of each s_{2i} and simplifying we get

$$0 = s_{21} \frac{z_{1n}}{z_{12} z_{2n}} + \sum_{j=3}^{n-1} s_{2j} \frac{z_{jn}}{z_{j2} z_{2n}} .$$

- Above equation can be changed to

$$0 = s_{21} + \sum_{j=3}^{n-1} s_{2j} \frac{z_{jn} z_{12}}{z_{j2} z_{1n}}$$

which is nothing, but the cross ratio identity we have discussed.

Example: $\frac{1}{z_{12}^3 z_{34}^3 z_{56}^3 z_{23} z_{45} z_{61}} = (123456) \frac{1}{z_{12}^2 z_{34}^2 z_{56}^2}$ having three double poles. To use the monodromy identity, we need to expand (123456) by others without pole s_{12}, s_{34}, s_{56}. One of such expansion is

$$
\begin{aligned}
(123456) = & \left(\left(\frac{-(s_{21}+s_{23})}{s_{12}} + \frac{-(s_{25}+s_{26})(s_{43}+s_{42})}{s_{12}s_{34}} \right) \frac{-(s_{56}+s_{54})}{s_{56}} \right. \\
& \left. + \frac{(s_{25}+s_{26})(s_{46}+s_{41})}{s_{12}s_{34}} \right) (132546) - \frac{s_{26}(s_{43}+s_{45})}{s_{21}s_{34}} (135426) \\
& + \left(\left(\frac{-(s_{21}+s_{23})}{s_{12}} + \frac{-(s_{25}+s_{26})(s_{43}+s_{42})}{s_{12}s_{34}} \right) \frac{(s_{53}+s_{51})}{s_{56}} + \frac{s_{26}(s_{41}+s_{46})}{s_{21}s_{34}} \right) (135246) \\
& + \left(\frac{(s_{25}+s_{26})s_{41}}{s_{12}s_{34}} \frac{(-)(s_{56}+s_{52})}{s_{56}} + \frac{s_{26}s_{41}}{s_{21}s_{34}} \right) (135264) + \frac{(s_{25}+s_{26})s_{41}}{s_{12}s_{34}} \frac{s_{54}}{s_{56}} (132645) \\
& + \left(\frac{-(s_{21}+s_{23})}{s_{12}} + \frac{-(s_{25}+s_{26})(s_{43}+s_{42})}{s_{12}s_{34}} \right) \frac{s_{51}}{s_{56}} (153246) + \frac{(s_{25}+s_{26})s_{41}}{s_{12}s_{34}} \frac{(s_{51}+s_{54})}{s_{56}} (153264)
\end{aligned}
$$

Remarks:

- Using the cross ratio identity we may give a proof of Feynman rules
- Having fast and analytic algorithm to write results for any CHY-integrands will open up a way to understand CHY-construction further

Thanks a lot for listening!!!