# Conformal Interfaces on Euclidean Sphere

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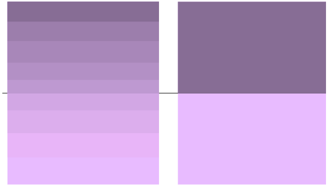
May 28, 2016

# Conformal Interfaces

- An interface refers to d-1 dimensional field theory system immersed inside a d-dimensional bulk.
- Interface CFT
  The bulk and the interfaces can be off-critical or critical separately. As change of parameters, the interface system may undergo variety of phase transitions.

- In this talk we would like to consider both the bulk and the interfaces are at criticality which allows us to use the AdS/CFT correspondence.



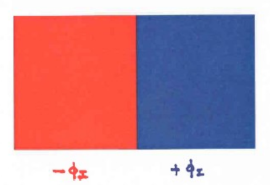
- This interface CFT is basically described by the Janus deformation of the bulk CFT. [ Bak-Gutperle-Hirano, 03, 07]
- Interface CFT
  The coupling costant dual to a marginal operator $O_d(x)$ jumps across the interface while keeping $SO(d,1)$ out of $SO(d+1,1)$ conformal symmetry.

$$\int d^d x \, [\, \mathcal{L}_{\rm CFT} + \phi_I \, \epsilon(x_1) O_d(x) \,]$$

Below we take $O_d$ to be the Lagrange density operator:

$$O_d(x) = \mathcal{L}(x) \ \leftrightarrow \ \phi(x) : \ \text{dilaton}$$

Begin with a planar interface of $R^{d-1}$ emmersed in the Euclidean $R^d$.



# Janus on a sphere

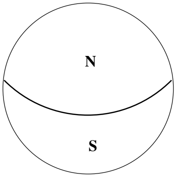
- By a conformal map, one can put the system on a Euclidean d sphere described by the metric

$$ds_d^2(\Omega) = r^2(d\theta^2 + \sin^2\theta\, ds^2_{S^{d-1}}(\omega))$$

where $\theta$ is the altitude coordinate ranged over $[0, \pi]$.

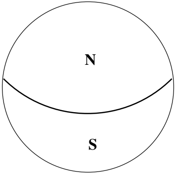
- The interface is located at the equator $\theta = \frac{\pi}{2}$ which is d-1 sphere.

- This will introduces localized degrees that couple to the northen and the southern hemispheres.



# Hybrid of $\text{CFT}_d$ and $\text{CFT}_{d-1}$

- Weyl and chiral symmetries are anomalous in even dimensions whereas they are intact in odd dimensional theories.
- Since our Janus system combines CFTs of even and odd dimensions at the same time, one finds always anomalies in one of bulk or interface.




- We will confirm this indeed!

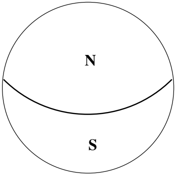
# Janus geometry

- AdS Einstein scalar system

$$I = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{g}\left(R + d(d-1) - g^{ab}\partial_a\phi\partial_b\phi\right)$$

- Einstein scalar equations:

$$R_{ab} + d\, g_{ab} = \partial_a\phi\partial_b\phi$$
$$\partial_a(\sqrt{g}g^{ab}\partial_b\phi) = 0$$




- This system can be embedded into IIB SUGRA in a consistent manner in the 3 and 5 dimensions. [ Bak-Gutperle-Hirano, 03, 07] Here we shall be mainly concerned with the three and four dimensional cases.

# The case without deformation: $\phi$ turned off

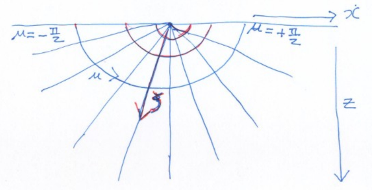
- We start with the Poincare patch coordinates

$$ds^2 = \frac{1}{z^2}[dz^2 + dt^2 + dx^2] = \frac{1}{\cos^2\mu}\left[d\mu^2 + \frac{dt^2 + d\xi^2}{\xi^2}\right]$$

Make the coordinate transformation

$$z = \xi \cos\mu\,, \quad x = \xi \sin\mu$$

where $-\pi/2 \le \mu \le \pi/2$.




- The total metric becomes an AdS2 sliced AdS3 Poincare patch coordinates.

- We now introduce another slicing coordinate $y$ defined by

$$dy = \frac{d\mu}{\cos\mu}$$

which is solved by

$$\cosh y = \frac{1}{\cos\mu}$$

This leads to the metric

$$ds^2 = dy^2 + f(y) ds^2_{AdS_2} = dy^2 + \cosh^2 y \, \frac{dt^2 + d\xi^2}{\xi^2}$$

where the slicing coordinate y is ranged over $(-\infty, \infty)$.

- We note that the $\mathrm{AdS}_2$ part can be replaced by any $\mathrm{AdS}_2$ metric that satisfies the $\mathrm{AdS}_d$ equation

$$\bar{R}_{pq} = -(d-1)\bar{g}_{pq}$$

with $d = 2$.

# Global Euclidean $AdS_3$

- We take the global Euclidean $AdS_2$ metric given by

$$ds^2_{M_2} = \frac{1}{\cos^2\lambda}\left[d\lambda^2 + \sin^2\lambda\, d\phi^2\right]$$

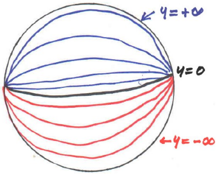
where $\lambda \in [0, \pi/2]$

- The full metric takes the form

$$ds^2 = dy^2 + f(y)ds^2_{M_2} = dy^2 + \cosh^2 y\,\frac{1}{\cos^2\lambda}\left[d\lambda^2 + \sin^2\lambda\, d\phi^2\right]$$

- One can see that y=0 has the shape of the $AdS_2$ disk.

- The slicing coordinate y runs in this manner.
- Positive y describes the upper part whereas the negative y corresponds to the lower part. The boundary has the shape of 2 sphere.

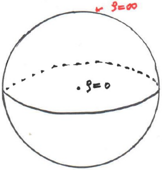



- Introducing the coordinates $\rho$ and $\theta$ by

$$\cosh\rho = \frac{\cosh y}{\cos\lambda}, \quad \cos\theta = \frac{\sinh y}{\sinh\rho}$$

- the above metric becomes

$$ds^2 = d\rho^2 + \sinh^2\rho\,\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$




- This is global Euclidean $AdS_3$ and the sphere shape of its boundary is rather clear.

# Janus deformation of global Euclidean $AdS_3$

- $AdS_2$ slicing ansatz

$$ds^2 = dy^2 + f(y) ds^2_{M_2}\,,\ \phi = \phi(y)$$

- This ansatz leads to ordinary differential equations,

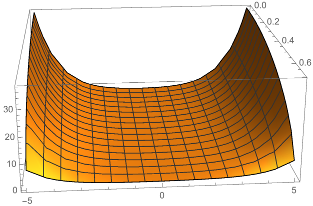
$$f' f' = 4f^2 - 4f + 4\gamma^2, \qquad \phi' = \frac{\gamma}{f}$$

where $\gamma$ is the deformation parameter related to the interface coefficient.

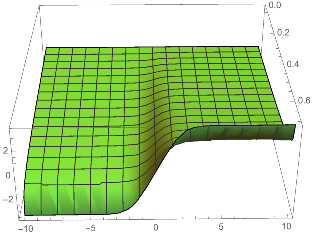
- One finds [ Bak-Gutperle-Hirano, 07]

$$f(y) = \frac{1}{2}(1 + \sqrt{1 - 2\gamma^2}\cosh 2y)$$

$$\phi(y) = \frac{1}{\sqrt{2}}\log\left(\frac{1 + \sqrt{1 - 2\gamma^2} + \sqrt{2}\gamma\tanh y}{1 + \sqrt{1 - 2\gamma^2} - \sqrt{2}\gamma\tanh y}\right)$$




scale factor

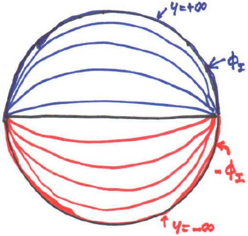



dilaton profile

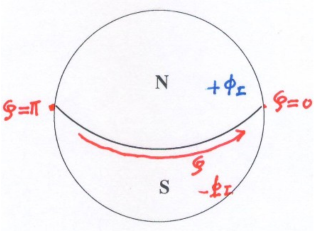
- $\gamma \in [0, 1/\sqrt{2}]$ and $-\phi_I \leq \phi(y) \leq \phi_I$ where

$$\phi_I = \frac{1}{\sqrt{2}}\text{arctanh}\sqrt{2}\gamma$$

# Shape of Janus3



cross sectional shape




boundary two sphere

- The explicit solutions for the higher dimensions:

$$\begin{aligned} ds^2_\pm &= \frac{\ell^2}{q^2_\pm}\left[\frac{dq^2_\pm}{P(q_\pm)} + ds^2_{M_d}\right] \\ \phi_\pm(q_\pm) &= \pm\gamma\int_{q_\pm}^{q_*} dx \frac{x^{d-1}}{\sqrt{P(x)}}, \end{aligned}$$

where where $P(x)$ is the dimension dependent polynomial

$$P(x) = 1 - x^2 + \frac{\gamma^2}{d(d-1)}x^{2d}$$

and $q_*$ denotes the smallest positive root of $P(x)$.

- We choose $M_d$ as the global Euclidean $AdS_d$ described by

$$ds^2_{M_d} = \frac{1}{\cos^2\lambda}[d\lambda^2 + \sin^2\lambda\, ds^2_{S^{d-1}}]$$

# Janus in general dimensions

- $q_\pm$ is ranged over $[0, q_*]$. To cover the entire space, the two corresponding patches should be joined at $q_\pm = q_*$.
- The explicit form is important here since we would like to carry out the integral in the evaluation of the partition functions below.
- Finally let us note the Janus geometry has the isometry SO(d,1) of $AdS_d$ space whereas the original $AdS_{d+1}$ vacuum geometry possesses SO(d+1,1) isometry.

# partition functions and dualities

- In this gravity side we evaluate the on-shell action corresponding to the free energy. $\rightarrow$ Divergent! We use the procedure of the so called holographic renormalization. [ de Haro-Soludukin-Skenderis, 01]
- Global AdS

$$ds^2 = d\rho^2 + \sinh^2\rho\, ds^2_{S^d}$$

Introduce the FG coordinate system

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}u^2}{u^2} + \frac{1}{u^2} h_{ij}(x,u^2)\,\mathrm{d}x^i\mathrm{d}x^j \\ &= \frac{\mathrm{d}u^2}{u^2} + \frac{1}{u^2}\left(1 - \frac{u^2}{4r^2}\right) r^2 ds^2_{S^d} \end{aligned}$$

where

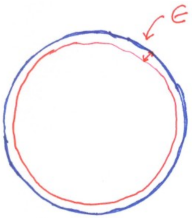
$$u = 2re^{-\rho}$$

# Regularization

- Regularization:

Introduce the cut-off surface near AdS boundary

$$u = \epsilon = 2r\delta$$




- The Einstein-Hilbert action with the Gibbons-Hawking term

$$I_{reg} = -\frac{1}{16\pi G}\int_{M_\epsilon} \mathrm{d}^{d+1}x\,\sqrt{g}\left[R - g^{ab}\partial_a\phi\partial_b\phi + d(d-1)\right] - \frac{1}{8\pi G}\int_{\partial M_\epsilon} \mathrm{d}^d x\,\sqrt{\gamma}K$$

- Carrying out the integral, the regularized action takes the form,

$$I_{\rm reg} = \frac{1}{16\pi G}\int \sqrt{h_B}\left(\frac{a_{(0)}}{\epsilon^d} + \frac{a_{(2)}}{\epsilon^{d-2}} + \cdots - 2\log(\epsilon)a_{(d)}\right) + O(\epsilon^0)$$

where the logarithmic contribution exists only when $d$ is even.

- In the holographic renormalization, we choose the counter-term as

$$I_{\rm ct} = -\frac{1}{16\pi G}\int \sqrt{h_B}\left(\frac{a_{(0)}}{\epsilon^d} + \frac{a_{(2)}}{\epsilon^{d-2}} + \cdots - 2\log(\epsilon)a_{(d)}\right)$$

- Adding together,

$$I_{ren} = \lim_{\epsilon\to 0}(I_{\rm reg} + I_{\rm ct})$$

- First, we keep the maximal remaining symmetry of SO(d+1) for our choice of the cut-off surface.

- The renormalized action is still dependent on the renormalization scheme one is adopting. For the match with the field theory computation, the renormalization schemes of the both sides have to be specified consistently.
- Second, the logarithmic term is related to the Weyl anomaly of even dimensional CFT.

$$I_{\rm reg} = \frac{1}{16\pi G}\int \sqrt{h_B}\left(\frac{a_{(0)}}{\epsilon^d} + \frac{a_{(2)}}{\epsilon^{d-2}} + \cdots - 2\log(\epsilon)a_{(d)}\right) + O(\epsilon^0)$$

where the logarithmic contribution exists only when $d$ is even.

- Thirdly later for the Janus deformation, the terms of remaining singular powers are also present:

$$b_{(1)}\epsilon^{-d+1} + b_{(3)}\epsilon^{-d+3} + \cdots$$

# Undeformed cases

- By a straightforward computation, one has

$$I_{\text{reg}} = \frac{\text{Vol}_{S^d}}{16\pi G}\frac{d}{2^{d-1}}\, C_d(\delta)$$

- For $d = 2$, one finds

$$C_2 = -\frac{1}{2}\left(\frac{1}{\delta^2} - \delta^2\right) + 2\log\delta$$

where, the logarithmic term is related to the Weyl anomaly of even dimensional CFT. This leads to the renormalized action

$$I_{\text{ren}} = -\frac{c}{3}\log 2r = -\frac{c}{3}\log \mu r$$

with the relation $c = 3/(2G)$.

- Thus the partition function is given by

$$Z_{\text{ren}} := \exp(-I_{\text{ren}}) = (r\mu)^{\frac{c}{3}}$$

Anomalous under $r \to (1+a)r$.

# d=3 and 4

- For $d = 3$, one has

$$C_3 = \frac{16}{3} - \frac{2}{3\delta^3} - \frac{2}{\delta} - 2\delta + \frac{2\delta^3}{3}$$

Thus the partition function is given by

$$Z_{\rm ren} = e^{-\frac{\pi}{2G}}$$

that agrees with the known result of $AdS_4 \times CP_3$. [ Marino 11]

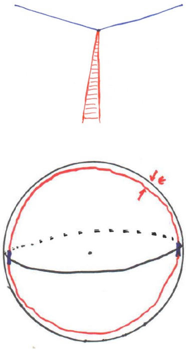
- For d=4, one finds

$$Z_{\rm ren} = (\mu r)^{-\frac{\pi}{2G}} = (\mu r)^{-N^2}$$

for the $\mathcal{N} = 4$ SU(N) SYM. [Pestun 07]

# Janus deformation

- One can only keep the SO(d) symmetry out of SO(d+1) symmetry of the boundary sphere $S^d$.



- Unfortunately FG coordinate is NOT globally defined with the Janus deformation.
- First we choose the cut off surface essentially the same as those of the undeformed geometry if one is at least infinitesimally away from the interface. For the remaining infinitesimal region we extend the above surface in a natural manner.

# $\mathrm{ICFT}_2$

- For d=2, with Janus solution, we find the bulk term together with the interface contribution

$$
\begin{aligned}
I_{\rm reg} &= I^0_{\rm reg} + \Delta I_{\rm reg} \\
&= \frac{1}{4G}\left[-\frac{1}{4}\frac{1}{\delta^2} + \log\delta + O(\delta)\right] \\
&+ \frac{1}{4G}\left[-\frac{2}{\delta}\alpha(\sqrt{1-2\gamma^2}) - \log\frac{1}{\sqrt{1-2\gamma^2}} + O(\delta)\right]
\end{aligned}
$$

where $\alpha(z)$ is

$$
\alpha(z) = \frac{\sqrt{1+z}}{\sqrt{2}}\left[\mathbf{K}\left(\frac{1-z}{1+z}\right) - \mathbf{E}\left(\frac{1-z}{1+z}\right)\right]
$$

# $\mathrm{ICFT}_2$

- With the minimal subtraction of the counter terms,

$$I_{\mathrm{ren}} = -\frac{1}{4G}\left[2\log(\mu r) + \frac{1}{2}\log\frac{1}{\sqrt{1-2\gamma^2}}\right]$$

- And the corresponding partition function becomes

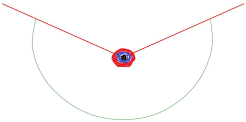
$$Z = Z_0\,\Delta Z = (\mu r)^{\frac{c}{3}}\left[\frac{1}{\sqrt{1-2\gamma^2}}\right]^{\frac{c}{6}}$$

- The interface contribution is scale invariant. This reflects the characteristic of odd dimensional CFT which preserves SO(2,1) conformal symmetries.

# Interface entropy

- $-\Delta I$ can be related to the interface entropy $S_I$ [Azeyanagi-Karch-Takayanagi -Thompson 08] by a conformal transformation

$$-\Delta I \rightarrow S_I = \ln(\cosh \sqrt{2}\phi_I)^{\frac{c}{6}}$$



- Janus black holes! [Bak-Gutperle -Janik 11]

# $\text{ICFT}_3$

- For d=3 Janus solution, the regulated action is evaluated as

$$
\begin{aligned}
I_{\text{reg}} &= I^0_{\text{reg}} + \Delta I_{\text{reg}} \\
&= \frac{\pi}{16G}\left(-\frac{1}{\delta^3} - \frac{3}{\delta} + 8 + O(\delta)\right) \\
&+ \gamma^2 \frac{\pi}{64G}\left(-\frac{7}{4\delta^2} + 4\log\delta - \frac{11}{2} + 4\log 2\right) + O(\gamma^4)
\end{aligned}
$$

- We see that the interface contribution has the structure of $\text{CFT}_2$ of two sphere. $\rightarrow$ Wely anomaly!

$$
\begin{aligned}
\Delta I_{\text{ren}} &= -\frac{c_{\text{eff}}(\gamma)}{3}\log r + d(\gamma) \\
c_{\text{eff}} &= \frac{3\pi\ell^2}{16G}\gamma^2 + O(\gamma^4) \\
d(\gamma) &= -\frac{11\pi\ell^2}{128\,G}\gamma^2 + O(\gamma^4)
\end{aligned}
$$

# Stress tensor

- Consider the undeformed CFT in two dimensions: Weyl trace anomaly dictates

$$\langle T^i{}_i \rangle_{\rm CFT} = \frac{c}{24\pi} R(h_B) = \frac{c}{12\pi r^2}$$

- On two sphere of radius $r$, one finds the stress tensor is proportional to the metric

$$\langle T_{ij} \rangle_{\rm CFT} = \frac{c}{24\pi r^2}\, h^B_{ij}$$

- For the Janus the holographic computation can be carried out infinitesimally away from the interface.

$$\langle T_{ij} \rangle_{\rm ICFT_2} = \frac{c}{24\pi r^2}\, h^B_{ij}$$

No interface contribution!

# Stress tensor of $\mathrm{ICFT}_3$

- Namely for d=3, again away from the interface, the standard holographic computation gives

$$\langle T_{ij} \rangle_{\mathrm{ICFT}_3} = 0 \qquad \text{if} \quad \theta \neq \frac{\pi}{2}$$

- The interface contribution can be identified from interface free energy which is scale dependent:

$$\begin{aligned} \Delta I_{\mathrm{ren}} &= -\frac{c_{\mathrm{eff}}(\gamma)}{3} \log r + d(\gamma) \\ c_{\mathrm{eff}} &= \frac{3\pi\ell^2}{16G}\gamma^2 + O(\gamma^4) \\ d(\gamma) &= -\frac{11\pi\ell^2}{128\,G}\gamma^2 + O(\gamma^4) \end{aligned} \Longrightarrow \begin{aligned} &\langle \Delta T_{\theta\theta} \rangle = \langle \Delta T_{\theta\alpha} \rangle = 0 \\ &\langle \Delta T_{\alpha\beta} \rangle = \frac{c_{\mathrm{eff}}}{24\pi r^2} h^B_{\alpha\beta}\, \delta\Big(\theta - \frac{\pi}{2}\Big) \end{aligned}$$

- The total energy momentum tensor is solely from the interface contribution.

# Check by Conformal Pertubation Theory

- Remember

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \phi_B(x)\,\mathcal{O}_\phi(x)$$

- Assuming

$$\langle \mathcal{O}_\phi \rangle_{\rm CFT} = 0$$

the correction to the free energy can be computed perturbatively

$$\Delta F = -\frac{1}{2!}\int \phi_B(x) \int \phi_B(x')\, \langle \mathcal{O}_\phi(x)\mathcal{O}_\phi(x')\rangle_{\rm CFT} + \cdots$$

where we use the information of the correlation function of the undeformed conformal field theory.

- One gets an agreement!

# Interface degrees and g-theorem

- In two dimensional CFT, extra number of ground states are produced by the presence of boundary or interface.
  $S_I = \ln g$ where $g$ is the number of the extra ground state.
  The g-theorem says [Affeck-Ludwig 91]

$$\frac{\mathrm{d}}{\mathrm{d}l} g(l) \leq 0$$

- If the RG flow is triggered by the operators localized on the boundary, one can prove this g-theorem. [Friedan-Konechny 03]
  However, if the RG flow is triggered by bulk operators, g function may either decrease or increase. [Green-Mulligan-Starr 08]

# Related Janus solutions

- Indeed if one consider two interfaces whose interface coefficients $\phi_I$ and $\phi'_I$, it is clear that [Bak-Min 14]

$$S_I(\phi_I, \phi'_I, l/r) \to S_I(\phi_I + \phi'_I)$$

as $l/r \to 0$.


- By conformal transformations one can construct the spherical shaped Janus solution on $R^d$ and Janus solution on $R \times S^{d-1}$: