Quantum Integrable Systems from Conformal Blocks

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Heng-Yu Chen
National Taiwan University

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Let us begin with a four point function of scalar conformal primary operator $\phi_i(x)$ with scale dimension Δ_i in d-dimensional CFTs:

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) >= \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b \frac{F(\mathrm{u},\mathrm{v})}{(x_{12}^2)^{\frac{(\Delta_1+\Delta_2)}{2}}(x_{34}^2)^{\frac{(\Delta_3+\Delta_4)}{2}}} \tag{1}$$

where $x_{ij} = x_i - x_j$ $a = \frac{\Delta_2-\Delta_1}{2}$, $b = \frac{\Delta_3-\Delta_4}{2}$ and $F(u, v)$ is a function of conformally invariant cross ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}).$$

We can decompose four point function further into contributions from the exchanged primary operators $\mathcal{O}_{\Delta,l}$ and its descendants:

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) >= \sum_{\{\mathcal{O}_{\Delta,l}\}} \lambda_{12\mathcal{O}_{\Delta,l}}\lambda_{34\mathcal{O}_{\Delta,l}}\mathcal{W}_{\mathcal{O}_{\Delta,l}}(x_i) \tag{2}$$

where $\mathcal{W}_{\mathcal{O}_{\Delta,l}}(x_i)$ is "*conformal partial wave*" and $\lambda_{12\mathcal{O}_{\Delta,l}}, \lambda_{34\mathcal{O}_{\Delta,l}}$ are "*OPE coefficients*".

Conformal invariance also allows us to fix $\mathcal{W}_{\Delta,l}(x_i)$ into:

$$\mathcal{W}_{\mathcal{O}_{\Delta,l}}(x_i) = \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b \frac{G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v})}{(x_{12}^2)^{\frac{(\Delta_1+\Delta_2)}{2}}(x_{34}^2)^{\frac{(\Delta_3+\Delta_4)}{2}}} \tag{3}$$

where $G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v})$ is the "*Conformal Block*" for $\mathcal{O}_{\Delta,l}$ family.

The four point functions need to satisfy crossing symmetry equation when e. g. $\phi_1(x_1) \leftrightarrow \phi_3(x_3)$:

$$\sum_{\{\mathcal{O}_{\Delta,l}\}} \lambda_{12\mathcal{O}_{\Delta,l}}\lambda_{34\mathcal{O}_{\Delta,l}} G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v}) = \frac{\mathrm{u}^{\frac{\Delta_1+\Delta_2}{2}}}{\mathrm{v}^{\frac{\Delta_2+\Delta_3}{2}}} \sum_{\{\mathcal{O}'_{\Delta,l}\}} \lambda_{32\mathcal{O}'_{\Delta,l}}\lambda_{14\mathcal{O}'_{\Delta,l}} G_{\mathcal{O}'_{\Delta,l}}(\mathrm{v},\mathrm{u}) \tag{4}$$

For unitary CFTs, if we know $G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v})$ exactly or at least some approximate forms, then assuming $\lambda_{12\mathcal{O}_{\Delta,l}}\lambda_{34\mathcal{O}_{\Delta,l}} \geq 0$, and start numerically putting bounds on spectrum of $\{\Delta_{\mathcal{O}}\}$.

Determining $G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v})$ for general four point functions remains difficult, one way is to consider "quadratic Casimir operator": [Dolan-Osborn 03, 11]

$$\hat{\mathbf{C}}_2 = \frac{1}{2} L^{AB} L_{AB} = \frac{1}{2}(L_1 + L_2)_{AB}(L_1 + L_2)^{AB} \tag{5}$$

where $L_{i,AB}$ is Lorentz generator in $d+2$-dimensional embedding space.

For scalar primaries, we can define following differential operators:

$$\begin{aligned} D_z^{(a,b,c)} &= z^2(1-z)\partial_z^2 - ((a+b+1)z^2 - cz)\partial_z - abz, \\ D_{\bar{z}}^{(a,b,c)} &= \bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - ((a+b+1)\bar{z}^2 - c\bar{z})\partial_{\bar{z}} - ab\bar{z}, \end{aligned} \tag{6}$$

$$\Delta_2^{(\varepsilon)}(a,b,c) = D_z^{(a,b,c)} + D_{\bar{z}}^{(a,b,c)} + 2\varepsilon \frac{z\bar{z}}{z-\bar{z}}((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}), \tag{7}$$

where $\varepsilon = \frac{d-2}{2}$. Setting $G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v}) = F^{(\varepsilon)}_{\lambda_+\lambda_-}(z,\bar{z})$, the action of $\hat{\mathbf{C}}_2$ is

$$\Delta_2^{(\varepsilon)}(a,b,0)\cdot F^{(\varepsilon)}_{\lambda_+\lambda_-}(z,\bar{z}) = \mathbf{c}_2(\lambda_+,\lambda_-)F^{(\varepsilon)}_{\lambda_+\lambda_-}(z,\bar{z}), \quad \lambda_\pm = \frac{\Delta \pm l}{2}. \tag{8}$$

In addition, we also have "quartic Casimir operator" :

$$\hat{\mathbf{C}}_4 = \frac{1}{2} L^{AB} L_{BC} L^{CD} L_{DA}. \tag{9}$$

The action of $\hat{\mathbf{C}}_4$ on primary scalars is also expressed as eigen-equation:

$$\Delta_4^{(\varepsilon)}(a, b, 0) \cdot F^{(\varepsilon)}_{\lambda_+\lambda_-}(z, \bar{z}) = \mathbf{c}_4(\lambda_+, \lambda_-) F^{(\varepsilon)}_{\lambda_+\lambda_-}(z, \bar{z}), \tag{10}$$

$$\Delta_4^{(\varepsilon)}(a, b, c) = \left[\frac{z\bar{z}}{z - \bar{z}}\right]^{2\epsilon} \left[D_z^{(a,b,c)} - D_{\bar{z}}^{(a,b,c)}\right] \left[\frac{z - \bar{z}}{z\bar{z}}\right]^{2\epsilon} \left[D_z^{(a,b,c)} - D_{\bar{z}}^{(a,b,c)}\right]. \tag{11}$$

The quadratic and quartic Casimir operators are by definition commuting:

$$[\hat{\mathbf{C}}_2, \hat{\mathbf{C}}_4] = 0 \Longrightarrow [\Delta_2^{(\varepsilon)}(a, b, c), \Delta_4^{(\varepsilon)}(a, b, c)] = 0. \tag{12}$$

Small Detour on Quantum Integrable Systems

A quantum integrable system can be constructed from classical one, provided the quantization condition [,] preserves commutators of $\{\hat{\mathrm{I}}_k\}$:

$$[\hat{\mathrm{I}}_j, \hat{\mathrm{I}}_k] = 0, \quad \frac{d\hat{\mathrm{I}}_k}{dt} = [\hat{\mathrm{I}}_k, \hat{\mathrm{H}}] = 0, \quad k = 1, \ldots, n.$$

The generator of $\{\hat{\mathrm{I}}_i\}$ is called "Monodromy matrix" $\hat{T}_{N,a}(\lambda)$, usually defined on a discretized lattice:

$$\hat{T}_{N,a}(\lambda) = \prod_{r=1}^{N} \hat{L}_{r,a}(\lambda), \quad \mathcal{H} \otimes V_a = \bigotimes_{r=1}^{N} \mathcal{H}_r \otimes V_a$$

such that Lax matrix $\hat{L}_{r,a}(\lambda)$ acts on local $\mathcal{H}_r$ and auxiliary space V_a.

For Heisenberg $XXX_{1/2}$ spin chain, $\mathcal{H} = \bigotimes_{r=1}^{N} \mathbb{C}_r^2$, $V_a = \mathbb{C}^2$

$$\hat{L}_{r,a}(\lambda) = \lambda \mathbf{1}_r \otimes \mathbf{1}_a + i \underbrace{(\vec{s}_r \cdot \vec{\sigma})}_{\text{on } \mathcal{H}_r} \otimes \vec{\sigma}_a = \hat{R}(\lambda - i/2) = (\lambda - i/2)\mathbf{1}_r \otimes \mathbf{1}_a + i\mathbb{P}_{r,a}$$

where $\mathbb{P}_{r,s}$ is the permutation operator $\mathbb{P}_{r,s} a_r \otimes b_s = b_r \otimes a_s$.

The generators of commuting conserved integrals of motions is:

$$t_N(\lambda) = \mathrm{Tr}_a\, \hat{T}_{N,a}(\lambda) = 2\lambda^N \mathbf{1}_{2\times 2} + \sum_{r=0}^{N-2} \hat{\mathrm{I}}_r \lambda^r, \quad [t_N(\lambda), t_N(\mu)] = 0,$$

spin component $\hat{S}_3$ completes N commuting quantum integrals of motion.

Interested in generalization of trigonometric "Calogero-Sutherland" spin systems: [Calogero, Sutherland 75]

$$\hat{\mathrm{H}}_{\mathrm{CS}} = -\sum_{i=1}^{N} (\partial_{q_i})^2 + \sum_{i\neq j} \frac{\beta(\beta - \mathrm{K}_{ij})}{\sin^2(q_i - q_j)} \Longleftarrow \text{Long range 2-body interaction,}$$

where K_{ij} is the long range permutation operator:

$$\mathrm{K}_{ij} | s_1, \dots, s_i, \dots, s_j \dots s_N >= | s_1, \dots, s_j, \dots, s_i, \dots, s_N >,$$

and $s_i = -l, -(l-1), \dots, l,\ l \in \frac{\mathbb{N}}{2}$.

The commuting integrals of motion are expressed in terms of the N commuting "Dunkl operators": [Dunkl 89]

$$\hat{\mathrm{J}}_i = i\partial_{q_i} + \beta \sum_{i \neq j} (1 - i \cot (q_i - q_j))\mathrm{K}_{ij} - 2\beta \sum_{j<i} \mathrm{K}_{ji},$$
$$[\hat{\mathrm{J}}_i, \hat{\mathrm{J}}_k] = 0, \; j, k = 1, \dots N$$

such that:

$$\hat{\mathrm{I}}_{2r} = -\sum_{i=1}^{N} \hat{\mathrm{J}}_i^{2r}, \quad [\hat{\mathrm{I}}_{2r}, \hat{\mathrm{I}}_{2s}] = 0, \quad r, s = 1, \dots N.$$

Importantly, the permutation symmetric eigenfunctions can be obtained analytically in terms of symmetric polynomials known as "Jack Polynomials", which are labeled by partition of non-negative integers $\lambda_1 \geq \lambda_2 \geq \dots \lambda_N \geq 0$, and we can study their various properties etc.

QIS from Conformal Blocks

Now we establish the following exact mapping between QIS and CFT:

[HYC-Qualls]

$$\left[\Delta_2^{(\varepsilon)}(a,b,c),\Delta_4^{(\varepsilon)}(a,b,c)\right]=0 \iff \left[\hat{\mathcal{I}}_2^{(\mathrm{BC}_2)},\hat{\mathcal{I}}_4^{(\mathrm{BC}_2)}\right]=0,$$

$$G_{\mathcal{O}_{\Delta,l}}(z,\bar{z}) \iff \psi^{(\epsilon)}_{\lambda_+\lambda_-}(u,\bar{u}).$$

The quantum integrable system here is given by following Hamiltonian:

$$\begin{aligned}\hat{\mathrm{H}}_{\mathrm{BC}_2} &= -\left(\frac{\partial^2}{\partial u^2}+\frac{\partial}{\partial \bar{u}^2}\right)+2\mathbf{a}\left(\frac{(\mathbf{a}-\mathrm{K}_{u\bar{u}})}{\sinh^2(u-\bar{u})}+\frac{(\mathbf{a}-\tilde{\mathrm{K}}_{u\bar{u}})}{\sinh^2(u+\bar{u})}\right)\\ &+ \left(\frac{\mathbf{b}(\mathbf{b}-\mathrm{K}_u)}{\sinh^2 u}-\frac{\mathbf{b}'(\mathbf{b}'-\mathrm{K}_u)}{\cosh^2 u}\right)+\left(\frac{\mathbf{b}(\mathbf{b}-\mathrm{K}_{\bar{u}})}{\sinh^2 \bar{u}}-\frac{\mathbf{b}'(\mathbf{b}'-\mathrm{K}_{\bar{u}})}{\cosh^2 \bar{u}}\right).\end{aligned}$$

This is called "Hyperbolic Calogero-Sutherland spin chain of BC_2, where

$$\begin{aligned}\text{Permutation} &: \mathrm{K}_{u\bar{u}}f(u,\bar{u})=f(\bar{u},u),\quad \tilde{\mathrm{K}}_{u\bar{u}}f(u,\bar{u})=f(-\bar{u},-u),\\ \text{Reflection} &: \mathrm{K}_u f(u,\bar{u})=f(-u,\bar{u}),\quad \mathrm{K}_{\bar{u}}f(u,\bar{u})=f(u,-\bar{u}).\end{aligned}$$

The commuting Dunkl operators are given explicitly by: [Finkel et al 12]

$$\hat{\mathrm{J}}^{(\mathbf{a})}_{u} = \frac{\partial}{\partial u} - \left[\frac{\mathbf{b}}{\tanh\frac{u}{2}} + \frac{\mathbf{b}'}{\coth\frac{u}{2}}\right]\mathrm{K}_u - \mathbf{a}\left[\frac{\tilde{\mathrm{K}}_{u\bar{u}}}{\tanh\frac{u+\bar{u}}{2}} + \frac{\mathrm{K}_{u\bar{u}}}{\tanh\frac{u-\bar{u}}{2}}\right],$$

$$\hat{\mathrm{J}}^{(\mathbf{a})}_{\bar{u}} = \frac{\partial}{\partial \bar{u}} - \left[\frac{\mathbf{b}}{\tanh\frac{\bar{u}}{2}} + \frac{\mathbf{b}'}{\coth\frac{\bar{u}}{2}}\right]\mathrm{K}_{\bar{u}} - \mathbf{a}\left[\frac{\tilde{\mathrm{K}}_{u\bar{u}}}{\tanh\frac{u+\bar{u}}{2}} - \frac{\mathrm{K}_{u\bar{u}}}{\tanh\frac{u-\bar{u}}{2}}\right],$$

and there are two independent commuting integrals of motion:

$$\hat{\mathcal{I}}^{(\mathrm{BC}_2)}_2 = \hat{\mathrm{H}}_{\mathrm{BC}_2} = -\left(\hat{\mathrm{J}}^{(\mathbf{a})}_{u}\right)^2 - \left(\hat{\mathrm{J}}^{(\mathbf{a})}_{\bar{u}}\right)^2, \quad \hat{\mathcal{I}}^{(\mathrm{BC}_2)}_4 = -\left(\hat{\mathrm{J}}^{(\mathbf{a})}_{u}\right)^4 - \left(\hat{\mathrm{J}}^{(\mathbf{a})}_{\bar{u}}\right)^4. \tag{13}$$

The first step is to look for the appropriate commutator-preserving coordinate transformation: [Isachenkov-Schomerus, HYC-Qualls]

$$\Delta^{(\varepsilon)}_{2,4}(a,b,c) \longrightarrow \chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})\Delta^{(\varepsilon)}_{2,4}(a,b,c)\frac{1}{\chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})} \tag{14}$$

to relate CFT Casimirs to the commuting quantum integrals of motion.

The desired transformation is given by the following double-cover map:

$$z(u) = -\frac{1}{\sinh^2 u}, \quad \bar{z}(\bar{u}) = -\frac{1}{\sinh^2 \bar{u}},$$

$$\chi^{(\varepsilon)}_{a,b,c}(z(u), \bar{z}(\bar{u})) = \frac{[(1-z(u))(1-\bar{z}(\bar{u}))]^{\frac{a+b-c}{2}+\frac{1}{4}}}{[z(u)\bar{z}(\bar{u})]^{\frac{1-c}{2}}} \left[\frac{z(u)-\bar{z}(\bar{u})}{z(u)\bar{z}(\bar{u})}\right]^{\varepsilon} \tag{15}$$

Explicit computation then shows that:

$$\chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})\Delta^{(\varepsilon)}_2(a,b,c)\frac{1}{\chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})} = -\frac{1}{4}\mathcal{I}_2 = -\frac{1}{4}\mathrm{H}_{\mathrm{BC}_2},$$

$$\mathbf{a} = \varepsilon, \quad \mathbf{b} = (a-b)+\frac{1}{2}, \quad \mathbf{b}' = (a+b-c)+\frac{1}{2}. \tag{16}$$

Notice when $\varepsilon = 0$ or $\varepsilon = 1$, i. e. $d = 2$ or $d = 4$, pair-wise interactions both vanish (Pöschl-Teller). More generally we have the correspondence:

$$\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}) = \chi^{(\varepsilon)}_{a,b,c}(z(u),\bar{z}(\bar{u}))F^{(\varepsilon)}_{\lambda_+\lambda_-}(z(u),\bar{z}(\bar{u}))$$

The eigenfunction is manifestly symmetric under $u \leftrightarrow \bar{u}$.

The quartic Casimir $\Delta_4^{(\varepsilon)}(a,b,c)$ can also be expressed in terms of $\hat{\mathcal{I}}_2^{(\mathrm{BC}_2)}$ and $\hat{\mathcal{I}}_4^{(\mathrm{BC}_2)}$. First we can show that acting on $\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u})$:

$$\left(\mathrm{J}_u^{(0)}\right)^2-\left(\mathrm{J}_{\bar{u}}^{(0)}\right)^2=4\chi^{(0)}_{a,b,c}(z,\bar{z})\left[D_z^{(a,b,c)}-D_{\bar{z}}^{(a,b,c)}\right]\frac{1}{\chi^{(0)}_{a,b,c}(z,\bar{z})}. \quad (17)$$

Next define following combinations of Dunkl and permutation operators:

$$\hat{\mathrm{L}}^{(+)}_{u\bar{u}}=\hat{\mathrm{J}}^{(\varepsilon)}_u+\hat{\mathrm{J}}^{(\varepsilon)}_{\bar{u}}+2\epsilon\tilde{\mathrm{K}}_{u\bar{u}},\quad \hat{\mathrm{L}}^{(-)}_{u\bar{u}}=\hat{\mathrm{J}}^{(\varepsilon)}_u-\hat{\mathrm{J}}^{(\varepsilon)}_{\bar{u}}+2\epsilon\mathrm{K}_{u\bar{u}}$$

We can next show that:

$$\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u})=t^{(\varepsilon)}(z,\bar{z})\left[\left(\mathrm{J}_u^{(0)}\right)^2-\left(\mathrm{J}_{\bar{u}}^{(0)}\right)^2\right]\frac{1}{t^{(\varepsilon)}(z,\bar{z})}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}),$$

$$t^{(\varepsilon)}(z(u),\bar{z}(\bar{u}))=\left[\sinh(u+\bar{u})\sinh(u-\bar{u})\right]^{\varepsilon}. \quad (18)$$

Using the invariance of $\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u})$ under reflection and permutation.

While above expression is invariant under K_u and $\mathrm{K}_{\bar{u}}$, however crucially:

$$\mathrm{K}_{u\bar{u}}\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}) = -\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}),$$
$$\tilde{\mathrm{K}}_{u\bar{u}}\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}) = -\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}),$$

These properties in turn imply:

$$\begin{aligned}
&\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}\\
&= \frac{1}{t^{(\varepsilon)}(z,\bar{z})}\left[\left(\mathrm{J}^{(0)}_u\right)^2 - \left(\mathrm{J}^{(0)}_{\bar{u}}\right)^2\right] t^{(\varepsilon)}(z,\bar{z})\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}\\
&= \frac{1}{t^{(\varepsilon)}(z,\bar{z})}\left[\left(\hat{\mathrm{J}}^{(0)}_u\right)^2 - \left(\hat{\mathrm{J}}^{(0)}_{\bar{u}}\right)^2\right] t^{(\varepsilon)}(z,\bar{z})^2\left[\left(\hat{\mathrm{J}}^{(0)}_u\right)^2 - \left(\hat{\mathrm{J}}^{(0)}_{\bar{u}}\right)^2\right]\frac{\psi^{(\varepsilon)}_{\lambda_+\lambda_-}}{t^{(\epsilon)}(z,\bar{z})}.
\end{aligned}$$

This precisely equals to gauge-transformed $\Delta^{(\varepsilon)}_4(a,b,c)$ and expanding

$$\chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})\Delta^{(\varepsilon)}_4(a,b,c)\frac{1}{\chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})} = -\frac{1}{8}\mathcal{I}_4 + \frac{1}{16}\mathcal{I}_2^2 + \frac{\varepsilon^2}{2}\mathcal{I}_2 + \varepsilon^4. \quad (19)$$

Now we have the following correspondence:

$$\chi_{a,b,c}^{(\varepsilon)}(z,\bar{z})\left[\Delta_2^{(\varepsilon)}(a,b,c),\Delta_4^{(\varepsilon)}(a,b,c)\right]\frac{1}{\chi_{a,b,c}^{(\varepsilon)}(z,\bar{z})}=[\mathcal{I}_2,\mathcal{I}_4]=0. \quad (20)$$

Not only $\Delta_2^{(\varepsilon)}(a,b,c)$ *but also* $\Delta_4^{(\varepsilon)}(a,b,c)$ *can be mapped to commuting conserved integrals of motion of* BC_2 *Calogero-Sutherland spin system!*

Supersymmetric Generalization

The correspondence established appears to be more general, also extends to superconformal field theories (SCFT) with four and eight supercharges. This can be attributed to two observations:

1. Superconformal blocks are known to be linearly expanded in terms of conformal blocks. [Poland-Simmons-Duffin 11, Fitzpatrick et al 14, Khandker et al 14]
2. Non-supersymmetric conformal blocks enjoy certain "recurrence relations" relating different $\{G_{\mathcal{O}_{\Delta,l}}(z,\bar{z})\}$. [Dolan-Osborn 03, 11]

They yield multiplicative relations between super and conformal blocks. More specifically, for SCFT with four supercharges we have: [Bobev et al 15]

$$\mathcal{G}^{\mathcal{N}=1}_{\Delta,l}(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}} G^{\Delta_{12}-1,\Delta_{34}-1}_{\Delta+1,l}(z,\bar{z}), \tag{21}$$

where $\mathcal{G}^{\mathcal{N}=1}_{\Delta,l}(z,\bar{z})$ is the superconformal block of scalar primaries: $< \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4) >$ with constraint $\Delta_{1,3} = \frac{d-1}{2} q_{1,3}$.

We can interpret this $(z\bar{z})^{-1/2}$ factor as part of gauge transformation. This is confirmed from superconformal $\hat{\mathbf{C}}_2^{\mathcal{N}=1}$, whose action is:

$$\Delta_2^{(\varepsilon)}(a+1,b,1)\cdot\mathcal{F}^{\mathcal{N}=1}_{\lambda_+\lambda_-}(z,\bar{z})=c_2^{\mathcal{N}=1}(\lambda_+\lambda_-)(z,\bar{z}). \tag{22}$$

The transformed $\Delta_2^{(\varepsilon)}(a+1,b,1)$ and $\mathcal{F}^{\mathcal{N}=1}_{\lambda_+\lambda_-}(z,\bar{z})$ are identified with the Hamiltonian and eigenfunction of BC_2 CS system with shift in $(\mathbf{b},\mathbf{b}')$.

For SCFTs with eight supercharges, we consider four point function:
[Lemos-Liendo 15]

$$\langle\Phi_1(x_1)\bar{\Phi}_2(x_2)\Phi_3(x_3)\bar{\Phi}_4(x_4)\rangle \tag{23}$$

where the $U(1)_R$ charges $q_1=-q_4$ and $q_2=-q_3$. In 4d $\mathcal{N}=2$ SCFT, it is an eigenfunction of $\Delta_2^{(1)}(a,a+2,2)$ which can be transformed into BC_2 CS system Hamiltonian. We would like to conjecture this to be true for arbitrary d-dimensions.

A small byproduct is the action of quartic superconformal Casimir:

$$\hat{\mathbf{C}}_4^{\mathcal{N}=1}\equiv\Delta_4^{(\varepsilon)}(a+1,b,1). \tag{24}$$

Generalization to External Spins

When external operators carry space-time spins, we need to decompose correlation function into independent Lorentz tensor structures. In four dimensions, we consider *"seed conformal partial waves"* [Echeverri et al 16]

$$W^{\rm seed}_{\mathcal{O}_{l,l+p}}(x_i) = \langle \phi_1(x_1) F_2(x_2)^{(p,0)} \phi_3(x_3) \bar{F}_4^{(0,p)}(x_4) \rangle$$
$$= \left(\frac{x_{14}^2}{x_{24}^2}\right)^{\frac{\tau_{21}}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\tau_{34}}{2}} \frac{\sum_{e=0}^{p} G_e^{(p)}(\mathrm{u},\mathrm{v}) \mathbf{I}_{42}^{e} \mathbf{J}_{42,31}^{p-e}}{(x_{12}^2)^{\frac{(\tau_1+\tau_2)}{2}} (x_{34}^2)^{\frac{(\tau_3+\tau_4)}{2}}},$$

where $\tau_i = \Delta_i + \frac{l_i + \bar{l}_i}{2}$, $\mathbf{I}_{ij}$ and $\mathbf{J}_{ij,kl}$ are independent tensor structures. Now $W^{\rm seed}_{\mathcal{O}_{l,l+p}}(x_i)$ as whole remains an eigenfunction of $\hat{\mathbf{C}}_2$, however $\mathbf{I}_{ij}$ and $\mathbf{J}_{ij,kl}$ are permuted, $\{G_e^{(p)}(\mathrm{u},\mathrm{v})\}$ obey coupled PDE instead:

$$\left[\Delta_2^{(\frac{p+2}{2})}(a_e, b_e, c_e) + \frac{\mathcal{E}_e^p}{2}\right] G_e^{(p)} + A_e^p z\bar{z}\mathbf{L}(a_{e-1}) G_{e-1}^{(p)} + B_e \mathbf{L}(b_{e+1}) G_{e+1}^{(p)} = 0 \tag{25}$$

Thanks to the recurrence relation for hypergeometric functions, these coupled PDEs can be solved iteratively:

$$G_e^{(p)}(z,\bar{z}) = \left(\frac{z\bar{z}}{z-\bar{z}}\right)^{2p} \sum_{(m,n)\in Oct_e^{(p)}} c_{m,n}^e F^{a_e,b_e,c_e}_{\frac{\Delta+l+p/2}{2}+m,\frac{\Delta-l+p/2}{2}-(p+1)+n}(z,\bar{z}).$$

where non-vanishing constant $c_{m,n}^e$ distribute in octagon $Oct_e^{(p)}$.

Remarkably, we can perform analogous term-wise gauge transformation to map above into:

$$\psi_e^{(p)}(u,\bar{u}) = \frac{1}{[\sinh(u+\bar{u})\sinh(u-\bar{u})]^{2p}} \sum_{(m,n)\in Oct_e^{(p)}} c_{m,n}^e \psi^{a_e,b_e,c_e}_{\rho_1+m,\rho_2+n}(u,\bar{u}), \tag{26}$$

i. e. the eigenfunctions of BC_2 Calogero-Sutherland system form a complete basis for expanding other conformal blocks.
In 2d, similar linear expansion of spinning conformal blocks in terms of scalar conformal blocks also occurs. [Osborn 12]

Future Directions

- Exact eigenfunctions (Koornwinder polynomials) and construction of other conformal blocks as series expansion in arbitrary d-dimensions.
- Can we diagonalize spinning blocks now linearly expanded in terms of scalar conformal block/CS eigenfunctions?
- If so, what kind of quantum integrable systems do they correspond to? If at all? How do they relate to one another?
- Superconformal blocks and quantum integrable systems?
- Is this connection with quantum integrable system accidental? Via AdS/CFT, can we see similar structures in gravity side?
  2 points $\rightarrow$ 3 points $\rightarrow$ 4 points?