What really happens when gluons collide?

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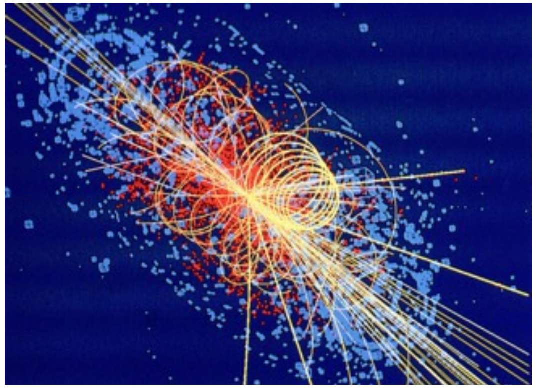
S-matrix in QFT

- Colliders at high energies need amplitudes of many gluons etc. (tree & loop level)



$gg \to gg \ldots g$

- Fundamental level: understanding of QFT incomplete; tensions with gravity
  deep structures & simplicity clearly seen in (perturbative) scattering amplitudes

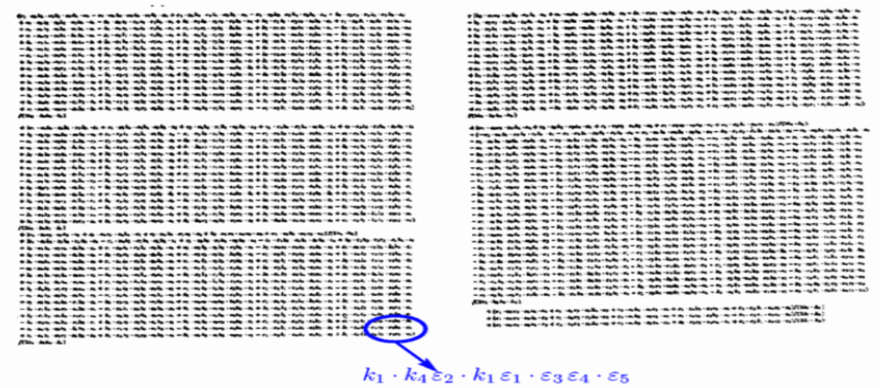
- Goal: new ideas & pictures of QFT & gravity from the study of their S-matrix

Feynman diagrams

- theoretical challenges: many diagrams, many many terms, gauge (non-)invariance

n-gluon scattering (tree)

n	4	5	6	7	8	9	10
# diagrams	4	25	220	2485	34300	559405	10525900



- Gluons: 2 states $h = \pm$, but manifest locality requires 4 states (huge redundancies)

- Much worse for graviton scattering (GR as EFT): redundancies from diff invariance

- *A prior* no reason to expect any simplicity or structures in the S-matrix

Parke-Taylor formula

- There is something going on: "Maximally-Helicity-Violating" amplitudes [Parke, Taylor, 86]

$$M_n(i^-, j^-) = \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle},$$

$$k_a^\mu = (\sigma^\mu)_{\alpha,\dot\alpha}\lambda_a^\alpha\tilde\lambda_a^{\dot\alpha}, \quad \epsilon_a^\pm = \ldots$$

$$\langle a\, b\rangle := \varepsilon_{\alpha,\beta}\,\lambda_a^\alpha\,\lambda_b^\beta, \quad [a\, b] := \varepsilon_{\dot\alpha,\dot\beta}\,\tilde\lambda_a^{\dot\alpha}\,\tilde\lambda_b^{\dot\beta}$$

spinor-helicity variables: "Chinese magic" [Xu, Zhang, Chang, 84,...]

classify gluon amps by # of - helicities: MHV, NMHV, N^2MHV, … , anti-MHV

color-decomposition into color-ordered (partial) amps: $\mathcal{M}_n = \sum_\pi \mathrm{Tr}(T^{I_{\pi(1)}}\cdots T^{I_{\pi(n)}})\; M_n[\pi]\,.$

- Led to 30 years of enormous progress on computing & understanding S-matrix

Twistor-string revolution

- Witten's twistor string theory: a worldsheet model for gluon tree amplitudes
  amps = string correlators fixed by a map from $\mathbb{C}\mathrm{P}^1$ to $\mathbb{C}\mathrm{P}^{3|4}$(twistor space) [Witten, 2003]

- Key observation: [Nair, 88] Parke-Taylor MHV amps = correlator on $\mathbb{C}\mathrm{P}^1$

$$\lambda_i^\alpha \sim (z_i, 1)\,, \qquad PT_n := \frac{1}{(z_1-z_2)\,(z_2-z_3)\cdots(z_n-z_1)}\,. \qquad j_A(z) j_B(z') = \frac{f_{AB}^C\, j_C}{z-z'} + \text{double poles} + \ldots$$

- $N^k MHV$ amplitude is the image of PT_n under the degree-(k+1) polynomial map!
  polarization dependence naturally encoded by maximal supersymmetry.

- Inspired numerous developments: twistor method, recursion relations, new progress of unitarity method, Grassmannian & on-shell diagrams, etc. etc.

Cachazo-He-Yuan formulation

- Witten's twistor string very special: d=4 N=4 super Yang-Mills theory
  - no supersymmetry? any spacetime dimension?
  - general theories: gravity, Yang-Mills, standard model, effective field theories?
  - generalizations to loop level?

- CHY formulation: scattering of massless particles in any dimension [CHY 2013]
  - *compact formulas* for amplitudes of gluons, gravitons, fermions, scalars, etc.
  - *manifest* gauge (diff) invariance, double-copy relations, soft theorems, etc.
  - *string-theory origin*: QFT amps as CFT correlators → loops from higher genus

Scattering equations

$$E_a := \sum_{b=1, b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}, \qquad a = 1, 2, \ldots, n$$

[CHY 2013]

- simplest "derivation": saddle point equations of Koba-Nielson factor [Gross, Mende]
- conditions for a map from Riemann sphere to the null cone to exist: $P(\sigma)^2 = 0$.
- determine locations of the n punctures in terms of the n null momenta (kinematics)
- universal, independent of theories
  the "kinetic part" of CHY formulation

Scattering equations

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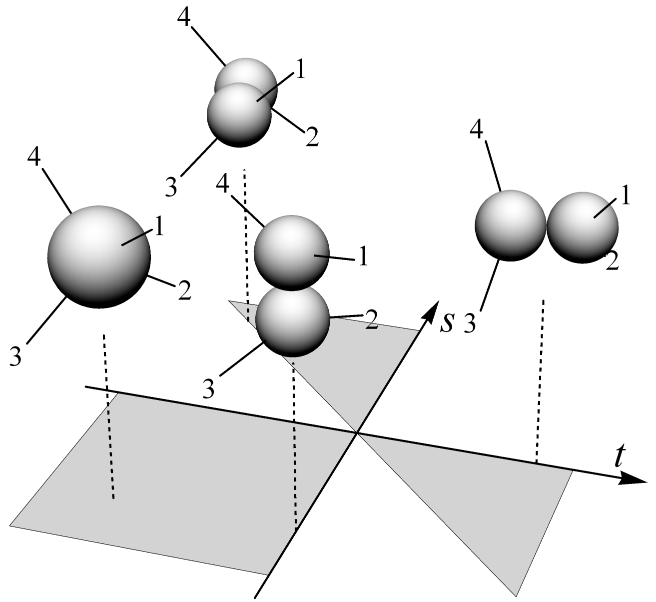


- kinematic space of n massless particles ↔ moduli space of n-punctured Riemann spheres

- map physical singularities (poles) to singularities of the moduli space (degenerations)

- Riemann spheres know lots of physics (locality & unitarity)

$$\{\sigma_2, \sigma_3, \sigma_4\} = \{0, 1, \infty\} \qquad \sigma_1 = -\frac{s_{12}}{s_{14}}$$

CHY formulas

$$M_n = \underbrace{\int \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)}_{d\mu_n} \mathcal{I}(\{k,\epsilon,\sigma\}) = \sum_{\{\sigma\}\in\text{solns.}} \frac{\mathcal{I}(\{k,\epsilon,\sigma\})}{J(\{\sigma\})}$$

- Amplitude as an integral over moduli space localized by scattering equations, or equivalently a sum over the solutions, with certain "CHY integrand" [CHY 2013]

- n-3 integrals with n-3 delta functions; J=det'$(\partial E/\partial\sigma)$ is the Jacobian

- $SL(2,\mathbb{C})$ symmetry: $\sigma_a \to \frac{\alpha\sigma_a+\beta}{\gamma\sigma_a+\delta}$, $E_a \to (\gamma\sigma_a+\delta)^2 E_a$ fix $\sigma_i, \sigma_j, \sigma_k$, $n-3$ variables; remove E_r, E_s, E_t, $n-3$ equations

$$\text{SL}(2,\mathbb{C}) : \sigma_a \to \frac{\alpha\,\sigma_a+\beta}{\gamma\,\sigma_a+\delta}, \quad \mathcal{I} \xrightarrow{SL(2,\mathbb{C})} \mathcal{I}\prod_{a=1}^{n}(\gamma\,\sigma_a+\delta)^4$$

CHY formulas

- Non-trivial polynomial eqs: (n-3)! solutions [CHY; Dolan, Goddard] $\rightarrow$ (n-3)! "virtual amps"
  In practice, evaluate the integral without solving equations [see Bo's talk]

- Key: a worldsheet picture for general massless particle scattering
  "interacting" via n-punctured Riemann spheres with certain correlators

- Tree amps = image of such correlators = sum of virtual amplitudes
  closed-formula for all amplitudes in a theory: Lagrangian, FD's etc. emergent

- Task: find "dynamic part", i.e. CHY integrands (correlators) for various QFT's

Simplest CHY formulas

- Parke-Taylor factor can be a "half integrand": $PT_n \rightarrow \prod_{a=1}^{n}(\gamma\sigma_a + \delta)^2 PT_n$

$$\mathrm{PT}[\pi] := \frac{1}{(\sigma_{\pi(1)} - \sigma_{\pi(2)})\,(\sigma_{\pi(2)} - \sigma_{\pi(3)}) \cdots (\sigma_{\pi(n)} - \sigma_{\pi(1)})}$$

- Simplest integrand: two copies of Parke-Taylor's with two orderings

$$m[\pi|\rho] := \int \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)\; PT[\pi]\; PT[\rho]\,.$$

- What does it compute? *trivalent scalar Feynman diagrams* (i.e. propagators!)

Scalar diagrams and ϕ^3 theory

- Sum of trivalent scalar Feynman diagrams consistent with both orderings:

$$m[\pi|\rho] = \sum_{g \in T(\pi) \cap T(\rho)} \prod_{e \in E(g)} \frac{1}{P_e^2}$$

- These are "double-partial amplitudes" of a bi-adjoint ϕ^3 theory:

$$\mathcal{L}_{\phi^3} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{3!} f^{I\,J\,K} f^{I'\,J'\,K'}\ \phi^{I\,I'} \phi^{J\,J'} \phi^{K\,K'}$$

$$M_n^{\phi^3} = \sum_{\pi,\rho} \mathrm{Tr}(T^{I_{\pi(1)}} \cdots T^{I_{\pi(n)}})\ \mathrm{Tr}(T^{I_{\rho(1)}} \cdots T^{I_{\rho(n)}})\ m[\pi|\rho]$$

Gluon scatterings from CHY

- What about gluons? need a new ingredient (half integrand) for polarizations.

- Inspired by the correlator of open-string vertex operators: using scattering equations, the correlator simplified to the Pfaffain of a simple matrix

$$\mathrm{Pf}'\Psi \sim \langle V^{(0)}(\sigma_1)\ldots V^{(-1)}(\sigma_i)\ldots V^{(-1)}(\sigma_j)\ldots V^{(0)}(\sigma_n)\rangle$$

- A formula for the complete S-matrix for any number of gluons in any dim:

$$M_n^{\mathrm{YM}}[\pi] = \int d\mu_n\, \mathrm{PT}[\pi]\, \mathrm{Pf}'\,\Psi$$

(gluon amps from "heterotic strings")

The Pfaffian

- The (reduced) Pfaffian of a $2n \times 2n$ skew matrix Ψ, with four blocks

$$\mathrm{Pf}'\Psi := \frac{\mathrm{Pf}|\Psi|^{i,j}_{i,j}}{\sigma_{i,j}} \qquad A_{a,b} := \begin{cases} \frac{k_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases}, \quad B_{a,b} := \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases},$$

$$\Psi := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \qquad C_{a,b} := \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ -\sum_{c \neq a} C_{a,c} & a = b \end{cases}$$

- The Pfaffian is permutation invariant, multi-linear in polarizations, …

  most importantly gauge invariant on the support of scattering equations!

Gauge invariance of the Pfaffian

- Gauge invariance $\epsilon_a^\mu \sim \epsilon_a^\mu + \alpha k_a^\mu$ is realized in the most straightforward way:

$$
\left(\begin{array}{cc|cc}
0 & \cdots & \sum_{b=2}^{n} \frac{k_1 \cdot k_b}{\sigma_{1,b}} & \cdots \\
\frac{k_2 \cdot k_1}{\sigma_{2,1}} & \cdots & \frac{k_2 \cdot k_1}{\sigma_{2,1}} & \cdots \\
\vdots & & \vdots & \\
\frac{k_n \cdot k_1}{\sigma_{2,1}} & \cdots & \frac{k_n \cdot k_1}{\sigma_{2,1}} & \cdots \\
-\sum_{b=2}^{n} \frac{k_1 \cdot k_b}{\sigma_{1,b}} & \cdots & 0 & \cdots \\
\frac{\epsilon_2 \cdot k_1}{\sigma_{2,1}} & \cdots & \frac{\epsilon_2 \cdot k_1}{\sigma_{2,1}} & \cdots \\
\vdots & & \vdots & \\
\frac{\epsilon_n \cdot k_1}{\sigma_{2,1}} & \cdots & \frac{\epsilon_n \cdot k_1}{\sigma_{2,1}} & \cdots
\end{array}\right)
$$

- with the replacement, the Pfaffian vanishes by scattering equations
- gauge invariance of string correlator
- (n-3)! virtual amplitudes individually gauge invariant (has all symmetries)!

Graviton scattering from CHY

- $PT \times PT \rightarrow$ bi-adjoint scalars, $PT \times Pf$ $\rightarrow$Yang-Mills, how about Einstein gravity?
- Two copies of Pfaffians for have polarizations $\epsilon^\mu \epsilon'^\nu$ (graviton: $h^{\mu\nu} = \epsilon^\mu \epsilon^\nu$)

$$M_n^{h+B+\phi} = \int d\mu_n \,\mathrm{Pf}'\Psi(\epsilon)\,\mathrm{Pf}'\Psi(\epsilon') \longrightarrow M_n^{\mathrm{GR}} = \int d\mu_n \,\det{}'\,\Psi(\epsilon)$$ ("closed strings")

- For any number of gravitons in any dim (hidden simplicity of perturbative GR)
- Double-copy relations "$GR \sim YM \otimes YM$"; more precisely $GR = YM^2/\phi^3$

Diffeomorphism invariance

- Exactly the same argument for diff invariance: $\epsilon_a^\mu \sim \epsilon_a^\mu + \alpha k_a^\mu$

$$\left(\begin{array}{cc|cc}
0 & \cdots & \sum_{b=2}^{n} \frac{k_1\cdot k_b}{\sigma_{1,b}} & \cdots \\
\frac{k_2\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{k_2\cdot k_1}{\sigma_{2,1}} & \cdots \\
\vdots & & \vdots & \\
\frac{k_n\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{k_n\cdot k_1}{\sigma_{2,1}} & \cdots \\
-\sum_{b=2}^{n} \frac{k_1\cdot k_b}{\sigma_{1,b}} & \cdots & 0 & \cdots \\
\frac{\epsilon_2\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{\epsilon_2\cdot k_1}{\sigma_{2,1}} & \cdots \\
\vdots & & \vdots & \\
\frac{\epsilon_n\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{\epsilon_n\cdot k_1}{\sigma_{2,1}} & \cdots
\end{array}\right)$$

- nothing but the closed-string correlator
- closed-string=(open-string)^2
- virtual amplitudes: $GR = YM^2/\phi^3$

Kawai-Lewellen-Tye (KLT) relations

- First such double-copy relations discovered as (field-theory limit of) KLT relations:

$$M_n^{\text{closed}} = \sum_{\alpha,\beta} M_n^{\text{open}}[\alpha]\ \mathcal{S}^{\text{string}}[\alpha|\beta]\ M_n^{\text{open}}[\beta] \quad \Longrightarrow \quad M_n^{\text{GR}} = \sum_{\alpha,\beta} M_n^{\text{YM}}[\alpha]\ S[\alpha|\beta]\ M_n^{\text{YM}}[\beta]\,.$$

- How about $GR = YM^2/\phi^3$? KLT derived from inserting two PT's in CHY: $\boldsymbol{S} = \boldsymbol{m}^{-1}$

$$M_n = \int d\mu_n\, \mathcal{I}_L\, \mathcal{I}_R \quad \Longrightarrow \quad M_n = \sum_{\alpha,\beta} M_L[\alpha]\ m^{-1}[\alpha|\beta]\ M_R[\beta], \quad \text{for } M_{L(R)} := \int d\mu_n\, \text{PT}\, \mathcal{I}_{L(R)}\,.$$

- A general way of seeing double-copy relations: splitting a CHY formula into two.

Soft theorems from CHY

- Universal behavior of amplitudes when a gauge boson/graviton becomes soft

$$M_n^{\text{gauge}} \to \sum_{i=1}^{n-1} e_i \, \frac{\epsilon_n \cdot k_i}{k_n \cdot k_i} \, M_{n-1}^{\text{gauge}} + \mathcal{O}(1), \qquad M_n^{\text{GR}} \to \sum_{i=1}^{n-1} \frac{(\epsilon_n \cdot k_i)^2}{k_n \cdot k_i} \, M_{n-1}^{\text{GR}} + \mathcal{O}(1), \qquad \text{as } k_n \to 0$$

- Soft theorems become manifest in CHY: residue theorem for σ_n-integral

$$\text{Pf}'\Psi_n \to \Big(\sum_{i=1}^{n-1} \frac{\epsilon_n \cdot k_i}{\sigma_{n,i}}\Big)^2 \, \text{Pf}'\Psi_{n-1} \,, \quad \Longrightarrow \quad M_n^{\text{GR}} \to \oint \frac{d\sigma_n}{\sum_{i=1}^{n-1} \frac{k_n \cdot k_i}{\sigma_{n,i}}} \Big(\sum_{i=1}^{n-1} \frac{\epsilon_n \cdot k_i}{\sigma_{n,i}}\Big)^2 M_{n-1}^{\text{GR}} \,.$$

- Sub-leading soft theorems work as well. Connections to BMS symmetry [Strominger,...]

More theories

- Generate CHY formulas of new theories from old ones, e.g. dim reduction

  GR → Einstein-Maxwell, YM → YM-scalar, with the Pfaffian factorizes, e.g.

$$M_{n\,\gamma}^{\mathrm{EM}} = \int d\mu_n \,\mathrm{Pf}'A \;\mathrm{Pf}X \;\mathrm{Pf}'\Psi \,, \quad M_{n\,s}^{\mathrm{YMs}} = \int d\mu_n \,\mathrm{Pf}'A \;\mathrm{Pf}X \;\mathrm{PT}\,; \quad X_{a\,b} = \frac{\delta^{I_a\,I_b}}{\sigma_{a,b}}(1-\delta_{a,b})\,.$$

- A new operation to add non-abelian interactions leads to "direct sum" of theories Formulas in Einstein ⊕ Yang-Mills and YM ⊕ bi-adjoint scalar theories [CHY 2014]

- A new class: effective field theories with Goldstone bosons. What is special here is that in the soft scalar limit, they have enhanced "Adler's zero" (Yu-tin's talk) [CHY 2015]

More theories

- $M_n = \int d\mu_n \, (\mathrm{Pf}'A)^2 \, \mathrm{PT}\,,$ adjoint derivative-coupled scalars?

  All tree amplitudes in the chiral Lagrangian (NLSM) $\mathcal{L} = \mathrm{Tr}(\partial_\mu U^+ \partial^\mu U)$

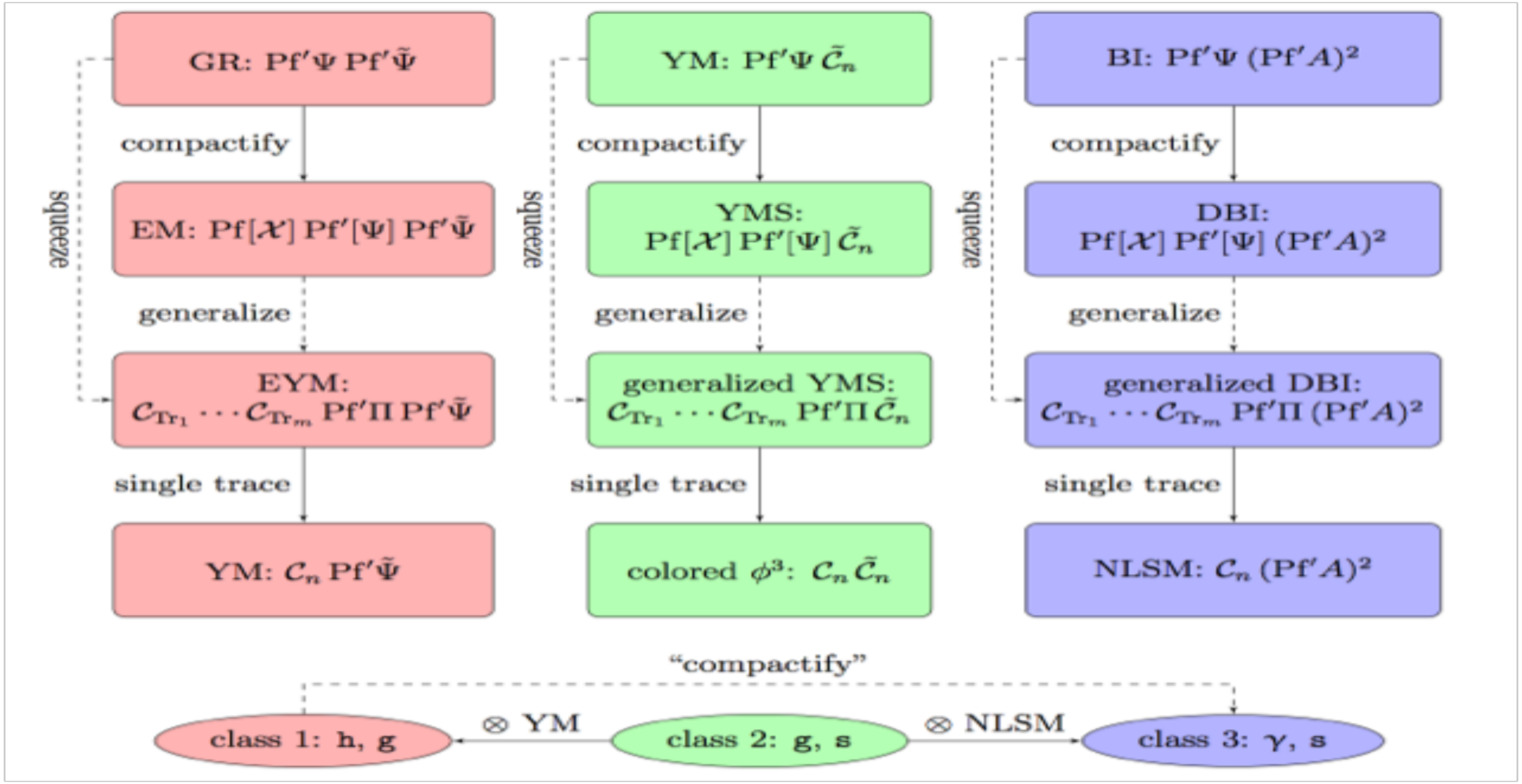
- $M_n = \int d\mu_n \, (\mathrm{Pf}'A)^2 \, \mathrm{Pf}'\Psi\,,$ higher-derivative-coupled photons?

  Born-Infeld theory (BI) & DBI by dim reduction $\mathcal{L} = \sqrt{-\det(\eta_{\mu\nu} - \ell F_{\mu\nu} - \ell^2 \partial_\mu \phi \partial_\nu \phi)}$

- a special Galileon theory (single scalar with many derivative: $M_n^{\mathrm{sGal}} = \int d\mu_n \, (\mathrm{Pf}'A)^4$

- KLT or "direct product": $BI \sim YM \otimes NLSM\,, DBI \sim YMs \otimes NLSM\,, sGal \sim NLSM^2$

A landscape of massless theories



Summary & outlook

- New picture: gluons (massless particles) scattering via punctures on a sphere.
  Suggest a weak-weak duality of QFT and string theory for S-matrix?

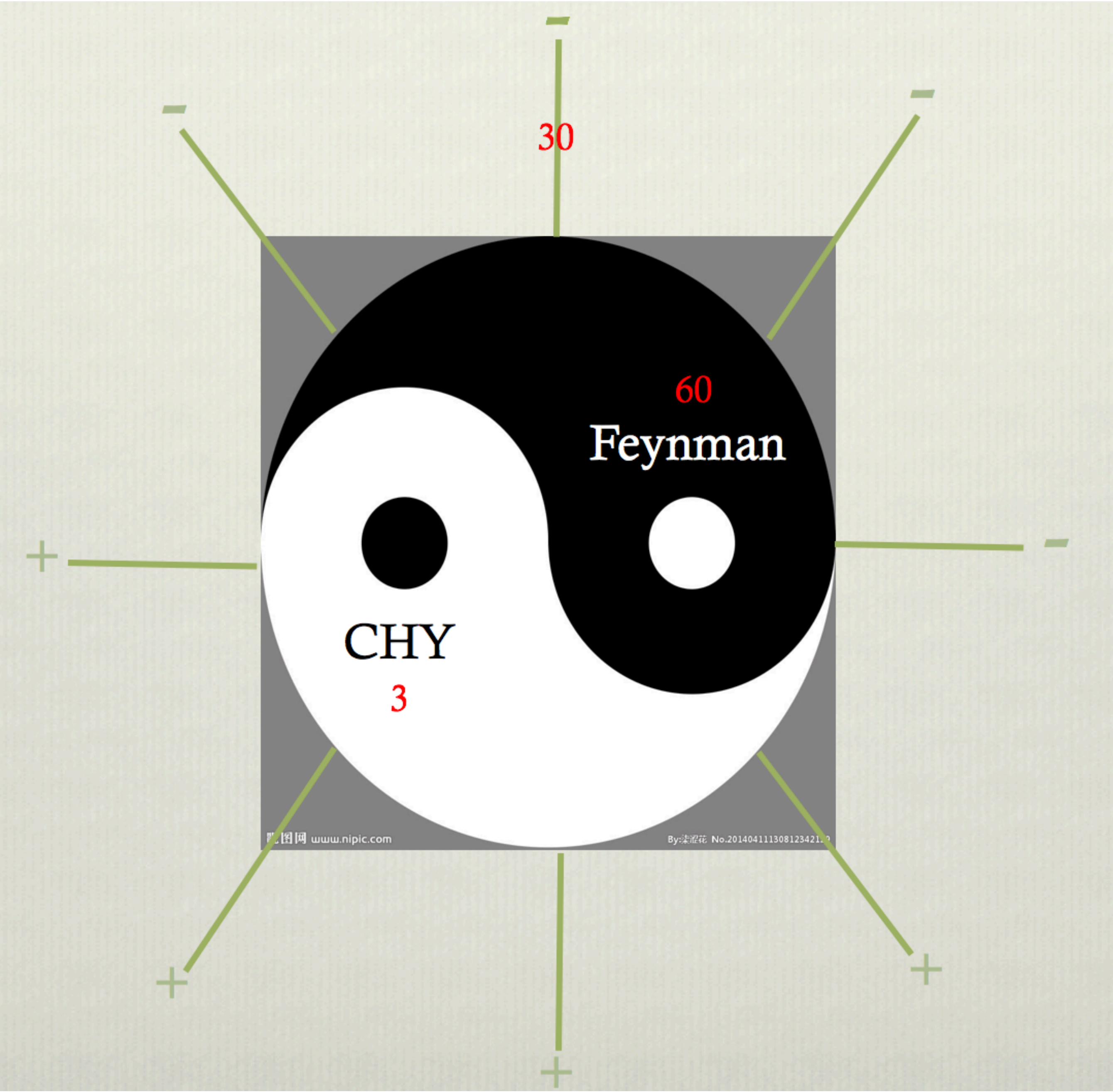
- Complimentary to FD's: (n-3)! virtual amplitudes with all symmetries manifest!

- Web of theories connected by operations e.g. $\oplus$ (interaction) & $\otimes$ (double-copy)

- Loop generalizations: higher-genus vs. higher-punctures [Adamo et al; Geyer et al; Feng et al; CHY…]

- Massive theories, fermions, Higgs etc. Scope of QFT's with natural CHY formula?

- S-matrix as representation theory, of (Poincare $\subset$ BMS $\subset$) some group?



taken from C.S. Lam's talk